Explicit free parametrization of the modified tetrahedron equation

2003 J. Phys. A: Math. Gen. 36 975

(http://iopscience.iop.org/0305-4470/36/4/309)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 975–998 PII: S0305-4470(03)52885-2

Explicit free parametrization of the modified tetrahedron equation

G von Gehlen[1], **S Pakuliak**[2] **and S Sergeev**[2,3]

[1] Physikalisches Institut der Universität Bonn, Nussallee 12, D-53115 Bonn, Germany
[2] Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Moscow region, Russia
[3] Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany

E-mail: gehlen@th.physik.uni-bonn.de, pakuliak@thsun1.jinr.ru, sergeev@mpim-bonn.mpg.de and sergeev@thsun1.jinr.ru

Received 27 August 2002, in final form 4 December 2002
Published 15 January 2003
Online at stacks.iop.org/JPhysA/36/975

Abstract
The modified tetrahedron equation (MTE) with affine Weyl quantum variables at the Nth root of unity is solved by a rational mapping operator which is obtained from the solution of a linear problem. We show that the solutions can be parametrized in terms of eight free parameters and 16 discrete phase choices, thus providing a broad starting point for the construction of three-dimensional integrable lattice models. The Fermat-curve points parametrizing the representation of the mapping operator in terms of cyclic functions are expressed in terms of the independent parameters. An explicit formula for the density factor of the MTE is derived. For the example $N = 2$ we write the MTE in full detail.

PACS numbers: 05.45.−a, 05.50.+q

Introduction

The Zamolodchikov tetrahedron equation is the condition [1] for the existence of a commuting set of layer-to-layer transfer matrices for three-dimensional lattice models, in much the same way as the Yang–Baxter equation is the analogous condition in the two-dimensional case. Only very few solutions to these very restrictive equations have been found [1, 2]. So various modified tetrahedron equations (MTE) have been studied to which more solutions can be obtained [3–8], still leading to commuting transfer matrices or generating functionals for conserved quantities.

In this paper we shall concentrate on a particular MTE proposed in [9, 10]. The quantum variables are elements from an ultra-local affine Weyl algebra attached to every vertex of a two-dimensional graph. Since we consider the Weyl parameter to be an Nth root of unity, the

Nth powers of the quantum variables form a classical system which determines the parameters of the quantum system, as has been considered in the discrete sine-Gordon model and other models recently, e.g. [11–14]. So the parameters of the eight $\mathbf{R}$ matrices appearing in the MTE are different, but related by functional mappings.

A linear problem discussed previously by one of us [9, 10], is used to determine the mapping which provides a multi-parameter solution to the MTE. The construction of a generating functional of the conserved quantities has been given in [9]. Here we concentrate on calculating the density factor of the MTE and to give a useful choice of the eight continuous parameters of the mapping.

The aim of studying these equations is at least twofold: first a (2+1)-dimensional integrable lattice model should emerge, and second, the MTE can be used by contraction [15] to construct new two-dimensional lattice models with parameters living on higher Riemann surfaces.

The paper is organized as follows. In section 1 we introduce the rational mapping in the affine Weyl space and show that if the Weyl parameter is a root of unity it splits into a matrix mapping and a functional mapping. The matrix mapping is realized in terms of cyclic functions which depend on points restricted to a Fermat curve. Section 2 discusses the modified tetrahedron equation and we calculate its weight function. In section 3 we focus on the parametrization in terms of line ratios, finding eight continuous parameters and analyse the phase ambiguities. For the specific case $N = 2$ in section 4 we show that the modified tetrahedron equations can be written quite explicitly and that of their 2^{12} matrix components there are 256 linearly independent equations. In section 5 we give a summary and mention future applications.

1. The rational mapping $\mathcal{R}$ in the space of a triple affine Weyl algebra

The central object of our considerations will be a mapping operator acting in the space of a triple Weyl affine algebra. We shall see that this mapping operator can be written as a superposition of a functional mapping and a finite-dimensional similarity transformation. It is the operator of this similarity transformation which will satisfy the MTE. Several interpretations are possible, e.g. as vertex Boltzmann weights (albeit not positive ones) of a three-dimensional lattice model. It will be a generalization of the Zamolodchikov–Bazhanov–Baxter [1, 2] Boltzmann weights in the Sergeev–Mangazeev–Stroganov [4] vertex formulation. The principle from which this mapping is obtained has been described in detail in [9]. It is a current conservation principle with a Baxter Z-invariance.

1.1. The linear problem

To set the framework we assign to each vertex j of a 2d graph the elements $\mathbf{u}_j$, $\mathbf{w}_j$ of an affine Weyl algebra at Weyl parameter q a root of unity:

$$\mathbf{u}_j \cdot \mathbf{w}_j = q\mathbf{w}_j \cdot \mathbf{u}_j \qquad q = \omega \stackrel{\text{def}}{=} \mathrm{e}^{2\pi\mathrm{i}/N} \qquad N \in \mathbb{Z} \quad N \geqslant 2. \tag{1}$$

Since q is a root of unity, $\mathbf{u}_j^N$ and $\mathbf{w}_j^N$ are centres of the Weyl algebra. We shall often represent the canonical pair $(\mathbf{u}_j, \mathbf{w}_j)$ by its action on a cyclic basis as unitary $N \times N$ matrices multiplied by complex parameters u_j, w_j, writing

$$\mathbf{u} = u\mathbf{x} \qquad \mathbf{w} = w\mathbf{z} \tag{2}$$

$$|\sigma\rangle \equiv |\sigma \bmod N\rangle \qquad \langle\sigma|\sigma'\rangle = \delta_{\sigma,\sigma'} \qquad \mathbf{x}|\sigma\rangle = |\sigma\rangle\omega^\sigma \qquad \mathbf{z}|\sigma\rangle = |\sigma + 1\rangle. \tag{3}$$

The centres are represented by numbers:

$$\mathbf{u}_j^N = u_j^N \qquad \mathbf{w}_j^N = w_j^N. \tag{4}$$

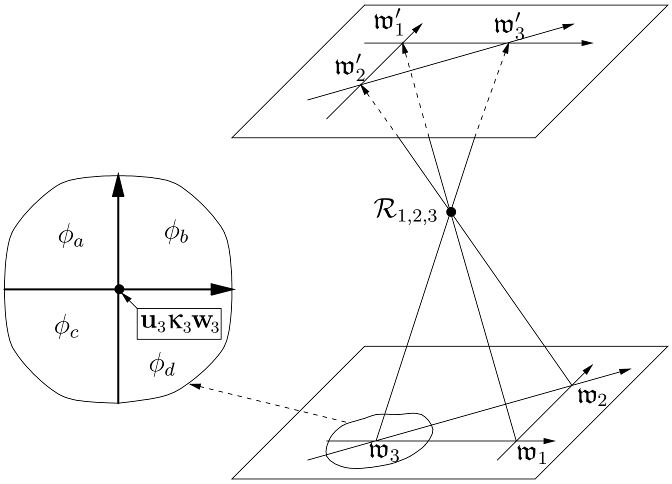

Figure 1. The linear problem for the vertex with associated Weyl pair $\mathbf{u}_3$, $\mathbf{w}_3$ and parameter κ_3 and the visualization of $\mathcal{R}_{1,2,3}$. The elements $\mathfrak{w}_1$, $\mathfrak{w}'_1$, etc of the ultra-local affine Weyl algebras are assigned to the vertices of two auxiliary two-dimensional lattices formed by the intersection of three straight lines with the auxiliary planes.

We define the ultra-local Weyl algebra $\mathfrak{W}^{\otimes\Delta}$ as the tensor product of Δ copies of Weyl pairs

$$\mathbf{u}_j = 1 \otimes 1 \otimes \cdots \otimes \underbrace{\mathbf{u}}_{\substack{j\text{th}\\ \text{place}}} \otimes \cdots \qquad \mathbf{w}_j = 1 \otimes 1 \otimes \cdots \otimes \underbrace{\mathbf{w}}_{\substack{j\text{th}\\ \text{place}}} \otimes \cdots. \tag{5}$$

Denote the four faces around the vertex j (in which the two oriented lines cross) clockwise by a, b, c, d as, e.g., shown on the left-hand side of figure 1 for the vertex $j = 3$. Imagine a current $\langle\phi|$ flowing out of the vertex into the four faces a, b, c, d, each face receiving a current $\langle\phi_s|$ $(s = a, b, c, d)$ according to the values of the Weyl variables at the vertex and a coupling constant κ_j (which may be different at each vertex):

$$\langle\phi| = \langle\phi_a| + \langle\phi_b| \cdot q^{1/2}\mathbf{u}_j + \langle\phi_c| \cdot \mathbf{w}_j + \langle\phi_d| \cdot \kappa_j \mathbf{u}_j \mathbf{w}_j. \tag{6}$$

Demanding that the total current flowing out of an internal vertex is zero: $\langle\phi| = 0$, and demanding also that the currents flowing into the outer faces of various graphs are independent of the internal structure of the graphs (this is a Z-invariance assumption), we get a condition for the equivalence of linear problems. The right-hand side of figure 1 shows two such linear problems, one in the bottom plane, another in the upper plane. The equivalence condition determines the mapping $\mathcal{R}_{1,2,3}$ between the lower $(\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3)$ and upper $(\mathfrak{w}'_1, \mathfrak{w}'_2, \mathfrak{w}'_3)$ triangle in figure 1 uniquely. The details of this calculation can be found in [9], in (7)–(9) we present the result.

For our case of interest $q = \omega$ it is convenient to choose the specific form (6) of the coefficients, which is unsymmetrical in a, b, c, d. There exists a fully symmetrical formulation of the linear problem valid at general q, still leading to a unique mapping $\mathcal{R}_{1,2,3}$ [9]. However, this will not be needed here.

1.2. The rational mapping $\mathcal{R}_{1,2,3}$

The solution of the equivalence problem of the linear current flows is the following rational mapping $\mathcal{R}$ acting on the ring of rational functions of the generators of the ultra-local Weyl algebra [9]. For any rational function Φ we define

$$(\mathcal{R}_{1,2,3} \circ \Phi)(\mathbf{u}_1, \mathbf{w}_1, \mathbf{u}_2, \mathbf{w}_2, \mathbf{u}_3, \mathbf{w}_3, \ldots) \stackrel{\text{def}}{=} \Phi(\mathbf{u}'_1, \mathbf{w}'_1, \mathbf{u}'_2, \mathbf{w}'_2, \mathbf{u}'_3, \mathbf{w}'_3, \ldots) \quad (7)$$

where on the right-hand side of (7) the $\mathbf{u}_\alpha$ and $\mathbf{w}_\alpha$ remain unchanged for all $\alpha \notin \{1, 2, 3\}$, and the primed elements are rational functions of $\mathbf{u}_1, \ldots, \mathbf{w}_3$, given by the definition

$$\begin{array}{lll} \mathbf{w}'_1 = \mathbf{w}_2 \cdot \Lambda_3 & \mathbf{w}'_2 = \Lambda_3^{-1} \cdot \mathbf{w}_1 & \mathbf{w}'_3 = \Lambda_2^{-1} \cdot \mathbf{u}_1^{-1} \\ \mathbf{u}'_1 = \Lambda_2^{-1} \cdot \mathbf{w}_3^{-1} & \mathbf{u}'_2 = \Lambda_1^{-1} \cdot \mathbf{u}_3 & \mathbf{u}'_3 = \mathbf{u}_2 \cdot \Lambda_1 \end{array} \quad (8)$$

where

$$\begin{aligned} \Lambda_1 &\equiv \mathbf{u}_1^{-1} \cdot \mathbf{u}_3 - q^{1/2}\mathbf{u}_1^{-1} \cdot \mathbf{w}_1 + \kappa_1 \mathbf{w}_1 \cdot \mathbf{u}_2^{-1} \\ \Lambda_2 &\equiv \frac{\kappa_1}{\kappa_2}\mathbf{u}_2^{-1} \cdot \mathbf{w}_3^{-1} + \frac{\kappa_3}{\kappa_2}\mathbf{u}_1^{-1} \cdot \mathbf{w}_2^{-1} - q^{-1/2}\frac{\kappa_1\kappa_3}{\kappa_2}\mathbf{u}_2^{-1} \cdot \mathbf{w}_2^{-1} \\ \Lambda_3 &\equiv \mathbf{w}_1 \cdot \mathbf{w}_3^{-1} - q^{1/2}\mathbf{u}_3 \cdot \mathbf{w}_3^{-1} + \kappa_3 \mathbf{w}_2^{-1} \cdot \mathbf{u}_3. \end{aligned} \quad (9)$$

$\kappa_1, \kappa_2, \kappa_3 \in \mathbb{C}$ are arbitrary extra parameters of the mapping $\mathcal{R}_{1,2,3}$. In this subsection q can be in a general position.

Note that the order of the factors on the right-hand sides of (8) does not matter. Each combination Λ_i $(i = 1, 2, 3)$ contains only elements of the Weyl algebra which commute with the other factors in the product. For example, Λ_3 does not have the Weyl operators $\mathbf{u}_1$ and $\mathbf{u}_2$. From (8) we see that the rational mapping $\mathcal{R}_{1,2,3}$ has the three invariants:

$$\mathbf{w}_1\mathbf{w}_2 \quad \mathbf{u}_2\mathbf{u}_3 \quad \mathbf{u}_1\mathbf{w}_3^{-1}. \quad (10)$$

This means that this mapping has the property that the products $\mathbf{u}_j^{-1}\mathbf{u}'_j$ and $\mathbf{w}_j^{-1}\mathbf{w}'_j$ for $j = 1, 2, 3$ (no summation over j) depend only on three operators which we denote by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{u} = \mathbf{w}_2^{-1}\mathbf{w}_3 \qquad \mathbf{v} = \mathbf{u}_1\mathbf{u}_2^{-1} \qquad \mathbf{w} = \mathbf{w}_1\mathbf{u}_3^{-1} \quad (11)$$

as one easily checks explicitly:

$$\begin{aligned} \left(\mathbf{w}_3^{-1}\mathbf{w}'_3\right)^{-1} &= \left(\mathbf{u}_1^{-1}\mathbf{u}'_1\right)^{-1} = \mathbf{u}_1\Lambda_2\mathbf{w}_3 = \frac{\kappa_1}{\kappa_2}\mathbf{v} + \frac{\kappa_3}{\kappa_2}\mathbf{u} - q^{-1/2}\frac{\kappa_1\kappa_3}{\kappa_2}\mathbf{v}\mathbf{u} \\ \left(\mathbf{u}_2^{-1}\mathbf{u}'_2\right)^{-1} &= \mathbf{u}'_3\mathbf{u}_3^{-1} = \mathbf{u}_3^{-1}\Lambda_1\mathbf{u}_2 = \mathbf{v}^{-1} + \kappa_1\mathbf{w} - q^{1/2}\mathbf{v}^{-1}\mathbf{w} \\ \mathbf{w}_1^{-1}\mathbf{w}'_1 &= \left(\mathbf{w}_2^{-1}\mathbf{w}'_2\right)^{-1} = \mathbf{w}_2\Lambda_3\mathbf{w}_1^{-1} = \mathbf{u}^{-1} + \kappa_1\mathbf{w}^{-1} - q^{1/2}\mathbf{w}^{-1}\mathbf{u}^{-1}. \end{aligned} \quad (12)$$

Observe that the three operators (11) form a triple Weyl algebra: $\mathbf{vu} = q\mathbf{uv}$, $\mathbf{vw} = q\mathbf{wv}$, $\mathbf{uw} = q\mathbf{wu}$, and also at each vertex j we can regard $\mathbf{u}_j, \mathbf{w}_j$ together with $\mathbf{v}_j \equiv \kappa_j\mathbf{u}_j\mathbf{w}_j$ as forming triple Weyl algebras.

The mapping $\mathcal{R}$ has the property (see [5, 7, 9, 10, 16] for details):

Proposition 1. *The invertible mapping $\mathcal{R}_{i,j,k}$ is an automorphism of $\mathfrak{W}^{\otimes\Delta}$.*

Remark 1. The proposition states that the rational mapping (8) is canonical, namely, it sends three copies of the ultra-local Weyl algebras into the same Weyl algebras.

Later, in section 2, proposition 3, we shall discuss the second crucial property of the mapping $\mathcal{R}$: it solves the tetrahedron equation (41).

1.3. Functional part at root of unity

In all the following we shall consider only the case that q is a root of unity (1) and use the unitary representation (3) of $\mathfrak{W}^{\otimes\Delta}$. In this representation each affine Weyl element $\mathbf{u}_j$ and $\mathbf{w}_j$ will contain one free parameter u_j, resp. w_j, as written in (2).

The basic fact is that at Weyl parameter root of unity any rational automorphism of the ultra-local Weyl algebra implies a rational mapping in the space of the Nth powers of the parameters of the representation [12–14]. In our case (8) it is easy to check that the mapping $\mathcal{R}_{1,2,3}$ implies

$$\begin{aligned} \mathbf{w}_1'^N &= w_2^N\Lambda_3^N \qquad & \mathbf{w}_2'^N &= \frac{w_1^N}{\Lambda_3^N} \qquad & \mathbf{w}_3'^N &= \frac{1}{\Lambda_2^N u_1^N} \\ \mathbf{u}_1'^N &= \frac{1}{\Lambda_2^N w_3^N} \qquad & \mathbf{u}_2'^N &= \frac{u_3^N}{\Lambda_1^N} \qquad & \mathbf{u}_3'^N &= u_2^N\Lambda_1^N \end{aligned} \tag{13}$$

where the Nth powers of the Λ_k are also numbers:

$$\begin{aligned} \Lambda_1^N &= u_1^{-N}u_3^N + u_1^{-N}w_1^N + \kappa_1^N w_1^N u_2^{-N} \\ \Lambda_2^N &= \frac{\kappa_1^N}{\kappa_2^N}u_2^{-N}w_3^{-N} + \frac{\kappa_3^N}{\kappa_2^N}u_1^{-N}w_2^{-N} + \frac{\kappa_1^N\kappa_3^N}{\kappa_2^N}u_2^{-N}w_2^{-N} \\ \Lambda_3^N &= w_1^N w_3^{-N} + u_3^N w_3^{-N} + \kappa_3^N w_2^{-N}u_3^N \end{aligned} \tag{14}$$

since for $q = \omega$ and $a, b \in \mathbb{C}$ one has $(a\mathbf{u} + b\mathbf{w})^N = (au)^N + (bw)^N$, using $\sum_{j\in\mathbb{Z}_N}\omega^j = 0$.

Definition 1. *The functional counterpart of the mapping $\mathcal{R}_{1,2,3}$ is the mapping $\mathcal{R}^{(f)}_{1,2,3}$, acting on the space of functions of the parameters u_j, w_j $(j = 1, 2, 3)$*

$$\big(\mathcal{R}^{(f)}_{1,2,3}\circ\phi\big)(u_1, w_1, u_2, w_2, u_3, w_3) \stackrel{\text{def}}{=} \phi(u_1', w_1', u_2', w_2', u_3', w_3') \tag{15}$$

where the primed variables are functions of the unprimed ones, defined via

$$u_1'^N = \mathbf{u}_1'^N \qquad w_1'^N = \mathbf{w}_1'^N \qquad etc \tag{16}$$

such that the u_j, w_j, u_j', w_j' satisfy

$$w_1'w_2' = w_1w_2 \qquad u_2'u_3' = u_2u_3 \qquad \frac{u_1'}{w_3'} = \frac{u_1}{w_3}. \tag{17}$$

The three free phases of the Nth roots are extra discrete parameters of $\mathcal{R}^{(f)}_{1,2,3}$.

We use the invariance of the three centres u_2u_3, u_1/w_3 and w_1w_2 to define three functions $\Gamma_1, \Gamma_2, \Gamma_3$:

$$\begin{aligned} \Gamma_1^N &= \frac{u_3'^N}{u_3^N} = \frac{u_2^N}{u_2'^N} = \big(u_3^{-1}\Lambda_1u_2\big)^N = \frac{u_2^N}{u_1^N} + \frac{w_1^Nu_2^N}{u_1^Nu_3^N} + \kappa_1^N\frac{w_1^N}{u_3^N} \\ \Gamma_2^N &= \frac{u_1^N}{u_1'^N} = \frac{w_3^N}{w_3'^N} = (w_3\Lambda_2u_1)^N = \frac{\kappa_1^Nu_1^N}{\kappa_2^Nu_2^N} + \frac{\kappa_3^Nw_3^N}{\kappa_2^Nw_2^N} + \frac{\kappa_1^N\kappa_3^Nu_1^Nw_3^N}{\kappa_2^Nu_2^Nw_2^N} \\ \Gamma_3^N &= \frac{w_1'^N}{w_1^N} = \frac{w_2^N}{w_2'^N} = \big(w_1^{-1}\Lambda_3w_2\big)^N = \frac{w_2^N}{w_3^N} + \frac{w_2^Nu_3^N}{w_1^Nw_3^N} + \kappa_3^N\frac{u_3^N}{w_1^N} \end{aligned} \tag{18}$$

so that, alternatively to using (13), (14), the functional mapping can be written as

$$\begin{aligned} \mathcal{R}^{(f)}\circ w_1 &= w_1\Gamma_3 \qquad & \mathcal{R}^{(f)}\circ w_2 &= \frac{w_2}{\Gamma_3} \qquad & \mathcal{R}^{(f)}\circ w_3 &= \frac{w_3}{\Gamma_2} \\ \mathcal{R}^{(f)}\circ u_1 &= \frac{u_1}{\Gamma_2} \qquad & \mathcal{R}^{(f)}\circ u_2 &= \frac{u_2}{\Gamma_1} \qquad & \mathcal{R}^{(f)}\circ u_3 &= u_3\Gamma_1. \end{aligned}$$

The Γ_j depend on three variables and the three constants κ_j. Their phases are arbitrary.

1.4. Matrix part at root of unity

Now we consider the matrix structure of $\mathcal{R}_{1,2,3}$ at q a root of unity. First of all, we define

$$\mathbf{x}'_1 = \frac{\mathbf{u}'_1}{u'_1} \qquad \mathbf{z}'_1 = \frac{\mathbf{w}'_1}{w'_1} \quad \text{etc.} \tag{19}$$

The normalization implies the conservation of the centres

$$\mathbf{x}'^N_1 = \mathbf{x}^N_1 = 1 \qquad \mathbf{z}'^N_1 = \mathbf{z}^N_1 = 1 \quad \text{etc.} \tag{20}$$

The rational mapping (8) for the set of matrices (19) has the form

$$\begin{aligned}
(\mathbf{x}'_1)^{-1} &= \frac{\kappa_1 u'_1}{\kappa_2 u_2}\mathbf{x}_2^{-1} + \frac{\kappa_3 u'_1 w_3}{\kappa_2 u_1 w_2}\mathbf{x}_1^{-1}\mathbf{z}_2^{-1}\mathbf{z}_3 - \omega^{1/2}\frac{\kappa_1\kappa_3 u'_1 w_3}{\kappa_2 u_2 w_2}\mathbf{x}_2^{-1}\mathbf{z}_2^{-1}\mathbf{z}_3\\
\mathbf{z}'_1 &= \frac{w_2 w_1}{w'_1 w_3}\mathbf{z}_1\mathbf{z}_2\mathbf{z}_3^{-1} - \omega^{1/2}\frac{w_2 u_3}{w'_1 w_3}\mathbf{z}_2\mathbf{x}_3\mathbf{z}_3^{-1} + \frac{\kappa_3}{w'_1}\mathbf{x}_3\\
(\mathbf{x}'_2)^{-1} &= \frac{u'_2}{u_1}\mathbf{x}_1^{-1} - \omega^{1/2}\frac{w_1 u'_2}{u_1 u_3}\mathbf{x}_1^{-1}\mathbf{z}_1\mathbf{x}_3^{-1} + \frac{\kappa_2 w_1 u'_2}{u_2 u_3}\mathbf{z}_1\mathbf{x}_2^{-1}\mathbf{x}_3^{-1}\\
(\mathbf{z}'_2)^{-1} &= \frac{w'_2}{w_3}\mathbf{z}_3^{-1} - \omega^{1/2}\frac{w'_2 u_3}{w_1 w_3}\mathbf{z}_1^{-1}\mathbf{x}_3\mathbf{z}_3^{-1} + \frac{\kappa_3 w'_2 u_3}{w_1 w_2}\mathbf{z}_1^{-1}\mathbf{z}_2^{-1}\mathbf{x}_3\\
\mathbf{x}'_3 &= \frac{u_2 u_3}{u_1 u'_3}\mathbf{x}_1^{-1}\mathbf{x}_2\mathbf{x}_3 - \omega^{1/2}\frac{w_1 u_2}{u_1 u'_3}\mathbf{x}_1^{-1}\mathbf{z}_1\mathbf{x}_2 + \frac{\kappa_2 w_1}{u'_3}\mathbf{z}_1\\
(\mathbf{z}'_3)^{-1} &= \frac{\kappa_1 u_1 w'_3}{\kappa_2 u_2 w_3}\mathbf{x}_1\mathbf{x}_2^{-1}\mathbf{z}_3^{-1} + \frac{\kappa_3 w'_3}{\kappa_2 w_2}\mathbf{z}_2^{-1} - \omega^{1/2}\frac{\kappa_1\kappa_3 u_1 w'_3}{\kappa_2 u_2 w_2}\mathbf{x}_1\mathbf{x}_2^{-1}\mathbf{z}_2^{-1}.
\end{aligned} \tag{21}$$

This mapping $\mathbf{x}_j, \mathbf{z}_j \mapsto \mathbf{x}'_j, \mathbf{z}'_j$, $j = 1, 2, 3$, is the basic example of a class of the canonical rational mappings of the ultra-local Weyl algebra. The following lemma establishes the uniqueness of the matrix structure of any such mapping.

Lemma 1. *Let $\mathbf{x}_j, \mathbf{z}_j$, $j = 1, \ldots, \Delta$ be a normalized finite-dimensional unitary basis (3) of the local Weyl algebra*

$$\mathbf{x}_i\mathbf{z}_j = q^{\delta_{i,j}}\mathbf{z}_j\mathbf{x}_i \qquad q^N = 1 \qquad \mathbf{x}_j^N = \mathbf{z}_j^N = 1. \tag{22}$$

Let $\mathcal{E}$: $\mathbf{x}_j, \mathbf{z}_j \mapsto \mathbf{x}'_j, \mathbf{z}'_j$ be an invertible canonical mapping in the space of rational functions of $\mathbf{x}_j, \mathbf{z}_j$ such that it conserves the centres

$$\mathbf{x}'^N_j = \mathbf{z}'^N_j = 1.$$

Then there exists a unique (up to a scalar multiplier) $N^\Delta \times N^\Delta$ matrix E such that for any Φ of equation (7):

$$\Phi(\mathbf{x}'_j, \mathbf{z}'_j) = E\,\Phi(\mathbf{x}_j, \mathbf{z}_j)E^{-1}. \tag{23}$$

Proof. The ring of the rational functions of $\mathbf{x}_j, \mathbf{z}_j$ at root of unity is the algebra of the polynomials of $\mathbf{x}_j, \mathbf{z}_j$ with $\mathbb{C}$-valued coefficients. Evidently this enveloping algebra is the complete algebra of $N^\Delta \times N^\Delta$ matrices. Since $\mathcal{E}$ is invertible, the envelope of $\mathbf{x}'_j, \mathbf{z}'_j$ is the same matrix algebra. Furthermore, since $\mathcal{E}$ is canonical and conserves the Nth powers of the Weyl elements, $\mathcal{E}$ is an automorphism of the matrix algebra. Finally, since the algebra of $N^\Delta \times N^\Delta$ matrices is the irreducible fundamental representation of the semi-simple algebra $\mathfrak{gl}(N^\Delta)$, any such automorphism is an internal one and may be realized by the unique matrix E of (23). $\square$

1.5. Matrix part of $\mathcal{R}_{1,2,3}$ at root of unity in terms of Fermat-curve cyclic functions $W_p(n)$

Due to lemma 1 there exists a unique (up to a scalar factor) $(N^3 \times N^3)$-dimensional matrix $\mathbf{R}_{1,2,3}$, such that

$$\mathbf{R}_{1,2,3}\mathbf{x}_1 = \mathbf{x}_1'\mathbf{R}_{1,2,3} \qquad \mathbf{R}_{1,2,3}\mathbf{z}_1 = \mathbf{z}_1'\mathbf{R}_{1,2,3} \quad \text{etc} \tag{24}$$

for (21).

The basis independent expression for $\mathbf{R}_{1,2,3}$ is not a useful object, and here we give the matrix elements of $\mathbf{R}_{1,2,3}$ in the basis (3) in terms of the Bazhanov–Baxter [2] cyclic functions $W_p(x)$ which we shall mainly use in this paper. We define

$$\frac{W_p(n)}{W_p(n-1)} = \frac{y}{1-\omega^n x} \qquad W_p(0) = 1 \tag{25}$$

where $n \in \mathbb{Z}_N$ and $p = (x, y)$ denotes a point on the Fermat curve

$$x^N + y^N = 1. \tag{26}$$

For $n > 0$ we have

$$W_p(n) = \prod_{\nu=1}^{n} \frac{y}{1-\omega^\nu x} \tag{27}$$

and generally $W_p(n+N) = W_p(n)$, because of $\prod_{\nu=0}^{N-1}(1-\omega^\nu x) = y^N$. One automorphism of the Fermat curve will be important for later calculations. Defining Op by

$$p = (x, y) \mapsto Op = (\omega^{-1}x^{-1}, \omega^{-1/2}x^{-1}y) \tag{28}$$

we have

$$W_p(n) = \frac{1}{W_{Op}(-n)\Phi(n)} \tag{29}$$

with

$$\Phi(n) = (-1)^n \omega^{n^2/2}. \tag{30}$$

At special points on the Fermat curve the $W_p(n)$ take simple values. Defining

$$q_0 = (0, 1) \qquad q_\infty = Oq_0 \qquad q_1 = (\omega^{-1}, 0) \tag{31}$$

we get

$$W_{q_0}(n) = 1 \qquad W_{q_\infty}(n) = \Phi^{-1}(n) \qquad 1/W_{q_1} = \delta_{n,0}. \tag{32}$$

We now express our conjugation matrix in terms of the functions $W_p(n)$:

Proposition 2. *In the basis (3) the matrix $\mathbf{R}_{1,2,3}$, solving the relations (21) and (24), has the following matrix elements:*

$$\langle i_1, i_2, i_3|\mathbf{R}_{1,2,3}|j_1, j_2, j_3\rangle \stackrel{\text{def}}{=} R_{i_1,i_2,i_3}^{j_1,j_2,j_3} = \delta_{i_2+i_3,j_2+j_3}\omega^{(j_1-i_1)j_3}\frac{W_{p_1}(i_2-i_1)W_{p_2}(j_2-j_1)}{W_{p_3}(j_2-i_1)W_{p_4}(i_2-j_1)} \tag{33}$$

where the x-coordinates of the four Fermat-curve points are connected by

$$x_1x_2 = \omega x_3x_4. \tag{34}$$

In the terms of the variables u_j, w_j, κ_j, $j = 1, 2, 3$, these points are defined by

$$x_1 = \frac{\omega^{-1/2}}{\kappa_1}\frac{u_2}{u_1} \qquad x_2 = \omega^{-1/2}\kappa_2\frac{u_2'}{u_1'} \qquad x_3 = \omega^{-1}\frac{u_2'}{u_1} \qquad x_4 = \omega^{-1}\frac{\kappa_2}{\kappa_1}\frac{u_2}{u_1'} \tag{35}$$

$$\frac{y_3}{y_1} = \kappa_1\frac{w_1}{u_3'} \qquad \frac{y_4}{y_1} = \omega^{-1/2}\kappa_3\frac{w_3}{w_2} \qquad \frac{y_3}{y_2} = \frac{w_2'}{w_3} \qquad \frac{y_4}{y_2} = \omega^{-1/2}\frac{\kappa_3}{\kappa_1}\frac{u_3'}{w_1'} \tag{36}$$

where the u'_j, w'_j and u_j, w_j are related by the functional transformation (15):

$$u'_j = \mathcal{R}^{(f)}_{1,2,3} \circ u_j \qquad w'_j = \mathcal{R}^{(f)}_{1,2,3} \circ w_j. \tag{37}$$

Proof. We shall give the proof, that (33) produces the rational mapping, for the first line of (21) only, the other equations follow analogously.

First observe that the matrix elements of the operator $\mathbf{R}_{1,2,3}$ satisfy several recurrent relations. In particular, we will need the recursion

$$R^{j_1,j_2,j_3}_{i_1,i_2+1,i_3-1} = R^{j_1,j_2,j_3}_{i_1,i_2,i_3} \cdot \frac{y_1}{y_4} \cdot \frac{1-\omega^{i_2-j_1+1}x_4}{1-\omega^{i_2-i_1+1}x_1}$$

which can be rewritten in the form

$$R^{j_1,j_2,j_3}_{i_1,i_2,i_3}\omega^{-j_1} = \frac{1}{\omega^{i_2+1}x_4}R^{j_1,j_2,j_3}_{i_1,i_2,i_3} + \frac{x_1y_4}{\omega^{i_1}y_1x_4}R^{j_1,j_2,j_3}_{i_1,i_2+1,i_3-1} - \frac{y_4}{\omega^{i_2+1}x_4y_1}R^{j_1,j_2,j_3}_{i_1,i_2+1,i_3-1}. \tag{38}$$

However, this recurrent relation is the matrix element $\langle i_1i_2i_3|\cdot|j_1j_2j_3\rangle$ of the operator equality

$$\mathbf{R}_{1,2,3}\cdot(\mathbf{x}'_1)^{-1} = \left(\frac{1}{\omega x_4}\mathbf{x}_2^{-1} + \frac{x_1y_4}{y_1x_4}\mathbf{x}_1^{-1}\mathbf{z}_2^{-1}\mathbf{z}_3 - \frac{y_4}{\omega x_4 y_1}\mathbf{x}_2^{-1}\mathbf{z}_2^{-1}\mathbf{z}_3\right)\cdot\mathbf{R}_{1,2,3}.$$

This coincides with the first line in (21), provided the identification (35) and (36) is valid for the Fermat points (x_1, y_1) and (x_4, y_4). □

Remark 2. **R** is a matrix function of three continuous parameters x_1, x_2, x_3 and three discrete parameters: the phases of y_1, y_2, y_3. Equivalently, one may use $\kappa_1\frac{u_1}{u_2}, \kappa_3\frac{w_3}{w_2}$ and $\frac{w_1}{u_3}$ as the continuous parameters and the phases of u'_1, u'_2, w'_1 as the discrete parameters. Formulae (35) and (36) establish the correspondence between these choices. We call the parametrization of $\mathbf{R}_{1,2,3}$ in terms of u_j, w_j, κ_j a 'free parametrization'.

Remark 3. Formulated in terms of mappings, the automorphism $\mathcal{R}_{1,2,3}$ of the ultra-local Weyl algebra at the root of unity is presented as the superposition of a pure functional mapping and the finite-dimensional similarity transformation:

$$\mathcal{R}_{1,2,3}\circ\Phi = \mathbf{R}_{1,2,3}\big(\mathcal{R}^{(f)}_{1,2,3}\circ\Phi\big)\mathbf{R}^{-1}_{1,2,3}. \tag{39}$$

2. The modified tetrahedron equation

For three-dimensional integrable spin models the tetrahedron equation (TE) plays the role which the Yang–Baxter equation has for two-dimensional integrable spin models. The TE provides the commutativity of the so-called layer-to-layer transfer matrices. In our case, where the dynamical variables form an affine Weyl algebra, we are able to define a more general equation: the modified tetrahedron equation (MTE), which provides the commutativity of more complicated transfer matrices, see e.g [3, 6]. In figure 2 we show a graphical image of the mappings leading to the tetrahedron equation (41): the sequence of mappings $Q_1 \to Q_2 \to Q_3 \to Q_4 \to Q_5$ gives the same Q_5 as $Q_1 \to Q_8 \to Q_7 \to Q_6 \to Q_5$.

Following equation (7) let us fix the following notation for the superposition of two mappings $\mathcal{A}$ and $\mathcal{B}$:

$$((\mathcal{A}\cdot\mathcal{B})\circ\Phi) \stackrel{\text{def}}{=} (\mathcal{A}\circ(\mathcal{B}\circ\Phi)). \tag{40}$$

Due to the uniqueness of the mapping $\mathcal{R}$ discussed in section 1.2, we arrive at

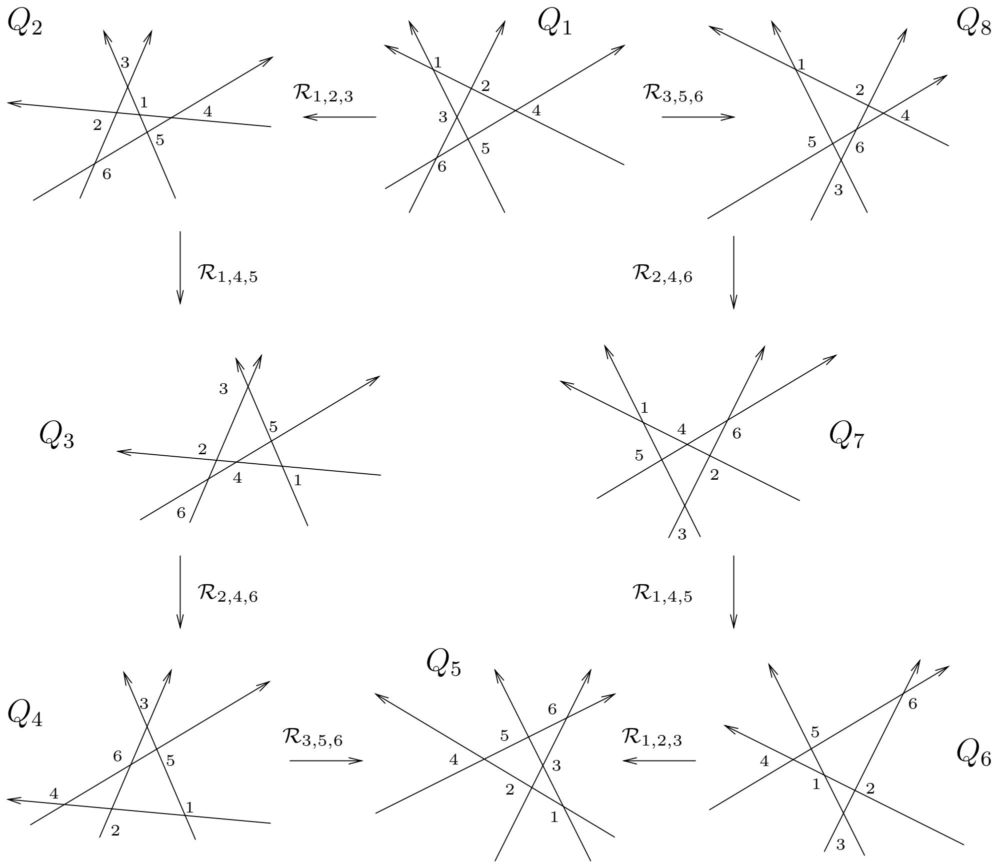

Figure 2. Graphical image of the two equivalent ways of transforming the four-line-graph ('quadrilateral') Q_1 into graph Q_5, which leads to the tetrahedron equation. Observe that each graph contains only two triangles which can be transformed by a mapping $\mathcal{R}$. In graph Q_1 either the line 124 can be moved downwards through the point 3 (leading to graph Q_2), or the line 456 can be moved upwards through point 3 (leading to Q_8). Both the left-hand and right-hand sequences of four transformations lead to the same graph Q_5.

Proposition 3. *The mapping $\mathcal{R}$ solves the tetrahedron equation:*

$$\mathcal{R}_{123} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{356} = \mathcal{R}_{356} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{123} \tag{41}$$

acting on the space of the 12 affine Weyl elements $\mathbf{u}_1, \mathbf{w}_1, \mathbf{u}_2, \mathbf{w}_2, \ldots, \mathbf{w}_5, \mathbf{u}_6, \mathbf{w}_6$.

Since, as has been discussed in section 1.3, any rational automorphism of the ultra-local Weyl algebra implies a rational mapping in the space of *N*th powers of the parameters of the representation, it is a direct consequence of (41) that the $\mathcal{R}^{(f)}_{i,j,k}$ of (15) solve the tetrahedron equation with the variables u_j^N, w_j^N, $j = 1, \ldots, 8$:

$$\mathcal{R}^{(f)}_{123} \cdot \mathcal{R}^{(f)}_{145} \cdot \mathcal{R}^{(f)}_{246} \cdot \mathcal{R}^{(f)}_{356} = \mathcal{R}^{(f)}_{356} \cdot \mathcal{R}^{(f)}_{246} \cdot \mathcal{R}^{(f)}_{145} \cdot \mathcal{R}^{(f)}_{123}. \tag{42}$$

We want to get this *functional* tetrahedron equation not only for the *N*th powers of the variables but for the variables u_j and w_j directly. However, when taking the *N*th roots, not all phases of the u_j, w_j can be chosen independently. In section 3.2 we shall show explicitly how to make an independent choice of phases.

Once an appropriate choice of phases has been made, we obtain the *functional* tetrahedron equation on the variables u_j, w_j. Now using (39) the functional tetrahedron equation can be cancelled between the two sides of (41), and we are left with the *modified tetrahedron equation* for the *finite-dimensional* **R** matrices:

$$\begin{aligned}\mathbf{R}_{1,2,3}\cdot\left(\mathcal{R}^{(f)}_{1,2,3}\circ\mathbf{R}_{1,4,5}\right)\cdot\left(\mathcal{R}^{(f)}_{1,2,3}\mathcal{R}^{(f)}_{1,4,5}\circ\mathbf{R}_{2,4,6}\right)\cdot\left(\mathcal{R}^{(f)}_{1,2,3}\mathcal{R}^{(f)}_{1,4,5}\mathcal{R}^{(f)}_{2,4,6}\circ\mathbf{R}_{3,5,6}\right)\\ \sim\mathbf{R}_{3,5,6}\cdot\left(\mathcal{R}^{(f)}_{3,5,6}\circ\mathbf{R}_{2,4,6}\right)\cdot\left(\mathcal{R}^{(f)}_{3,5,6}\mathcal{R}^{(f)}_{2,4,6}\circ\mathbf{R}_{1,4,5}\right)\cdot\left(\mathcal{R}^{(f)}_{3,5,6}\mathcal{R}^{(f)}_{2,4,6}\mathcal{R}^{(f)}_{1,4,5}\circ\mathbf{R}_{1,2,3}\right).\end{aligned}\tag{43}$$

Observe that due to the cancellation of the functional tetrahedron equation there is no $\mathcal{R}^{(f)}_{3,5,6}$ on the left-hand side of (43) and no $\mathcal{R}^{(f)}_{1,2,3}$ on the right-hand side. Because of the uniqueness of the mapping shown in lemma 1, the left- and right-hand sides of (43) may differ only by a scalar factor, which arises when we pass from the equivalence of the mappings to the equality of the matrices. So, in matrix element notation the MTE reads

$$\begin{aligned}&\sum_{j_1\cdots j_6}(R^{(1)})^{j_1,j_2,j_3}_{i_1,i_2,i_3}(R^{(2)})^{k_1,j_4,j_5}_{j_1,i_4,i_5}(R^{(3)})^{k_2,k_4,j_6}_{j_2,j_4,i_6}(R^{(4)})^{k_3,k_5,k_6}_{j_3,j_5,j_6}\\ &\qquad=\rho\sum_{j_1\cdots j_6}(R^{(8)})^{j_3,j_5,j_6}_{i_3,i_5,i_6}(R^{(7)})^{j_2,j_4,k_6}_{i_2,i_4,j_6}(R^{(6)})^{j_1,k_4,k_5}_{i_1,j_4,j_5}(R^{(5)})^{k_1,k_2,k_3}_{j_1,j_2,j_3}\end{aligned}\tag{44}$$

where $R^{(1)}$ corresponds to $\mathbf{R}_{1,2,3}$, $R^{(2)}$ to $\mathcal{R}^{(f)}{}_{1,2,3}\circ\mathbf{R}_{1,4,5}$, etc. Here ρ is the scalar factor. The matrix elements of each $R^{(j)}$ as functions of the Fermat-curve parameters $x_i^{(j)}$, $y_i^{(j)}$ are given by proposition 3 with functional mappings applied as shown in (43). The N^3th power of the scalar factor in (44) is

$$\rho^{N^3}=\frac{\det\mathbf{R}^{(1)}\det\mathbf{R}^{(2)}\det\mathbf{R}^{(3)}\det\mathbf{R}^{(4)}}{\det\mathbf{R}^{(8)}\det\mathbf{R}^{(7)}\det\mathbf{R}^{(6)}\det\mathbf{R}^{(5)}}\tag{45}$$

and this can be obtained from the determinant of one single matrix $\mathbf{R}_{1,2,3}$ just by substituting the respective coordinates.

2.1. *Calculation of the determinant of* $\mathbf{R}_{1,2,3}$

We use the representation (33) to find a closed expression for $\det\mathbf{R}_{1,2,3}$. The numerator term $W_{p_2}(j_2-j_1)$ is diagonal, so it just contributes a factor $\left(\prod_n W_{p_2}(n)\right)^{N^2}$ to the determinant. For later convenience, we treat the other numerator factor $W_{p_1}(i_2-i_1)$ of (33) differently: using the Fermat-curve automorphism (28) we write

$$W_{p_1}(n)=\frac{1}{W_{Op_1}(-n)\Phi(n)}.$$

$W_{Op_1}(i_1-i_2)$ is diagonal and its determinant is trivially calculated. We combine the factor $\Phi(i_2-i_1)$ with the two non-diagonal terms W_{p_3}, W_{p_4} and write

$$\det\langle i_1,i_2,i_3|\mathbf{R}|j_1,j_2,j_3\rangle=\left(\prod_{n=0}^{N-1}\frac{W_{p_2}(n)}{W_{Op_1}(n)}\right)^{N^2}\det\frac{\delta_{i_2+i_3,j_2+j_3}\,\omega^{(j_1-i_1)j_3}}{\Phi(i_2-i_1)W_{p_3}(j_2-i_1)W_{p_4}(i_2-j_1)}.\tag{46}$$

We now calculate the determinant on the right-hand side of (46) from its finite Fourier transform in the indices i_1 and j_1. So we define and evaluate

$$
\begin{aligned}
\langle i_1, i_2, i_3|\mathbf{R}'_{1,2,3}|j_1, j_2, j_3\rangle &= \delta_{i_2+i_3,j_2+j_3}\frac{1}{N}\sum_{a,b\in\mathbb{Z}_N}\omega^{i_1a-j_1b}\frac{\omega^{(b-a)j_3}}{\Phi(i_2-a)W_{p_3}(j_2-a)W_{p_4}(i_2-b)} \\
&= \delta_{i_2+i_3,j_2+j_3}\frac{1}{N}\sum_{a',b'}\omega^{i_1(j_2-a')-j_1(i_2-b')-(i_2-j_2)a'}\frac{\omega^{(i_2-j_2+a'-b')j_3}}{\Phi(i_2-j_2)\Phi(a')W_{p_3}(a')W_{p_4}(b')} \\
&= \delta_{i_2+i_3,j_2+j_3}\frac{\omega^{i_1j_2-i_2j_1+(j_3-i_3)j_3}}{\Phi(j_3-i_3)}\frac{1}{N}\sum_{a'}\frac{\omega^{(i_3-i_1)a'}}{\Phi(a')W_{p_3}(a')}\sum_{b'}\frac{\omega^{-(j_3-j_1)b'}}{W_{p_4}(b')}.
\end{aligned}
\tag{47}
$$

In the last line we have used the property

$$
\Phi(a+b) = \Phi(a)\Phi(b)\omega^{ab} \tag{48}
$$

which, supplying a factor $\omega^{(i_2-j_2)a'}$, decouples the a'- and b'-summations in the presence of $\delta_{i_2+i_3,j_2+j_3}$. The determinant of the first three factors of the last line of (47) will again be calculated from its Fourier transform. So we define and evaluate

$$
\begin{aligned}
\langle i_1, i_2, i_3|P|j_1, j_2, j_3\rangle &= \frac{1}{N^2}\sum_{c,d}\omega^{i_1c-j_1d}\delta_{i_2+i_3,j_2+j_3}\frac{\omega^{j_2c-i_2d+(j_3-i_3)j_3}}{\Phi(i_3)\Phi(j_3)}\omega^{i_3j_3} \\
&= \delta_{i_2+i_3,j_2+j_3}\delta_{i_1,-j_2}\delta_{j_1,-i_2}\frac{\Phi(j_3)}{\Phi(i_3)}
\end{aligned}
\tag{49}
$$

so that

$$
\det\langle i_1, i_2, i_3|P|j_1, j_2, j_3\rangle = \det\left\|\delta_{i_2+i_3,j_2+j_3}\delta_{i_1,-j_2}\delta_{j_1,-i_2}\right\| = (-1)^{N^2(N-1)/2}. \tag{50}
$$

Combining (46), (47), (49) and (50), our preliminary result is

$$
\det\langle i_1, i_2, i_3|\mathbf{R}_{1,2,3}|j_1, j_2, j_3\rangle
= \left((-1)^{(N-1)/2}\prod_k\frac{W_{p_2}(k)}{W_{Op_1}(k)}\prod_m\sum_a\frac{\omega^{ma}}{\Phi(a)W_{p_3}(a)}\prod_n\sum_b\frac{\omega^{nb}}{W_{p_4}(b)}\right)^{N^2}. \tag{51}
$$

We may simplify this formula by introducing the function $V(x)$ on the Fermat curve

$$
V(x) \stackrel{\text{def}}{=} \prod_{n=1}^{N-1}(1-\omega^{n+1}x)^n. \tag{52}
$$

Writing $(V(\omega^{-1}))^2$ as a product of N^2 factors of the form $(1-\omega^n)$ times powers of ω, we compute

$$
V(\omega^{-1}) = N^{N/2}\,\mathrm{e}^{\mathrm{i}\pi(N-1)(N-2)/12}. \tag{53}
$$

It is useful to note that

$$
\prod_n\sum_a\frac{\omega^{na}}{\Phi(a)} = V(\omega^{-1}) \qquad \prod_n\Phi(n) = \mathrm{e}^{-\pi\mathrm{i}(N^2-1)/6}. \tag{54}
$$

Writing out the factors of $(x^{N(N-1)/2}\,V(\omega^{-1}x^{-1}))\;V(x)$ and extracting several powers of ω, we obtain

$$
V(\omega^{-1}x^{-1}) = \frac{y^{N(N-1)}}{x^{N(N-1)/2}}\frac{\mathrm{e}^{\mathrm{i}\pi(N-1)(N-2)/6}}{V(x)}. \tag{55}
$$

We can express the terms involving p_1 and p_2 using $V(x)$:

$$
\prod_n W_{p_2}(n) = \frac{V(x_2)}{y_2^{N(N-1)/2}} \qquad \prod_n\frac{1}{W_{Op_1}} = \frac{V(x_1)}{y_1^{N(N-1)/2}}\,\mathrm{e}^{-\mathrm{i}\pi(N^2-1)/6}.
$$

In the ratio $V(x)/V(\omega x)$ many terms cancel, leading to

$$\frac{V(x)}{V(\omega x)} = \left(\frac{y}{1-\omega x}\right)^N. \tag{56}$$

This can be used to compute the p_4 term in (51). We define

$$F_{p=(x,y)} = \prod_n \sum_b \frac{\omega^{nb}}{W_p(b)}$$

which, using $W_{(\omega x,y)}(b) = (1-\omega x)y^{-1}W_{(x,y)}(b+1)$, is seen to satisfy

$$\frac{F_{(\omega x,y)}}{F_{(x,y)}} = \omega^{N(N-1)/2}\left(\frac{y}{1-\omega x}\right)^N \qquad F_{q_1} = 1 \qquad F_{q_\infty} = V^*(\omega^{-1}) \qquad F_{q_0} = 0$$

where q_0, q_∞ and q_1 are the three special Fermat points introduced in (31). This is solved by

$$F_{(x,y)} = (\omega x)^{N(N-1)/2}\frac{V(\omega^{-1})}{V(x)}.$$

Finally, we consider the p_3 term: Defining

$$G_{p=(x,y)} = \prod_n \sum_a \frac{\omega^{na}}{\Phi(a)W_p(a)}$$

which satisfies

$$\frac{G_{(\omega x,y)}}{G_{(x,y)}} = \left(\frac{y}{1-\omega x}\right)^N.$$

Using $G_{q_1} = 1$, $G_{q_\infty} = 0$, $G_{q_0} = V(\omega^{-1})$, we get $G_p = V(\omega^{-1})/V(x)$.

Inserting these results into (51), our final expression for $\det \mathbf{R}_{1,2,3}$ is

$$\det \mathbf{R} = N^{N^3}\left(\left(\frac{x_4}{y_1 y_2}\right)^{N(N-1)/2}\frac{V(x_1)V(x_2)}{V(x_3)V(x_4)}\right)^{N^2}. \tag{57}$$

The relation $x_1x_2 = \omega x_3x_4$ has not yet been used and is still to be imposed here. For $N = 2$ equation (57) gives

$$\det \mathbf{R} = 2^8\left(\frac{x_4}{y_1y_2}\frac{(1-x_1)(1-x_2)}{(1-x_3)(1-x_4)}\right)^4 = \left(4x_4\frac{y_1y_2}{y_3^2y_4^2}\frac{(1+x_3)(1+x_4)}{(1+x_1)(1+x_2)}\right)^4$$

in agreement with (77). Observe that, despite the quite similar appearance of W_{p_3} and W_{p_4} in (33), different phases make (57) unsymmetrical between x_3 and x_4, compare (77).

3. Parametrization of the Fermat points

Now the most important step follows: the parametrization of the Fermat points for each of the eight R matrices. Recall that $x_4^{(j)}$ is determined by the other three $x_i^{(j)}$ due to (34).

Writing repeatedly the parametrizations (35) and (36) for the eight R matrices, then applying repeatedly the functional mappings as written in (43), we get for the arguments appearing in (44):

$$\begin{aligned} &x_1^{(j)} = \frac{1}{\sqrt{\omega}}\,\Xi_{j1} \qquad x_2^{(j)} = \frac{1}{\sqrt{\omega}}\,\Xi_{j2} \qquad x_3^{(j)} = \frac{1}{\omega}\,\Xi_{j3} \qquad x_4^{(j)} = \frac{x_1^{(j)}x_2^{(j)}}{\omega x_3^{(j)}} \\ &y_{31}^{(j)} = \Upsilon_{j1} \qquad y_{41}^{(j)} = \frac{1}{\sqrt{\omega}}\Upsilon_{j2} \qquad y_{32}^{(j)} = \Upsilon_{j3} \qquad \text{where} \quad y_{ik}^{(j)} \equiv \frac{y_i^{(j)}}{y_k^{(j)}} \end{aligned} \tag{58}$$

where (not writing $\Xi_{j,4}$):

$$\Xi_{jk} = \begin{pmatrix} \frac{u_2^{(1)}}{\kappa_1 u_1^{(1)}} & \frac{\kappa_2 u_2^{(2)}}{u_1^{(2)}} & \frac{u_2^{(2)}}{u_1^{(1)}} \\ \frac{u_4^{(1)}}{\kappa_1 u_1^{(2)}} & \frac{\kappa_4 u_4^{(3)}}{u_1^{(5)}} & \frac{u_4^{(3)}}{u_1^{(2)}} \\ \frac{u_4^{(3)}}{\kappa_2 u_2^{(2)}} & \frac{\kappa_4 u_4^{(5)}}{u_2^{(5)}} & \frac{u_4^{(5)}}{u_2^{(2)}} \\ \frac{u_5^{(3)}}{\kappa_3 u_3^{(2)}} & \frac{\kappa_5 u_5^{(5)}}{u_3^{(5)}} & \frac{u_5^{(5)}}{u_3^{(2)}} \\ \frac{u_2^{(7)}}{\kappa_1 u_1^{(6)}} & \frac{\kappa_2 u_2^{(5)}}{u_1^{(5)}} & \frac{u_2^{(5)}}{u_1^{(6)}} \\ \frac{u_4^{(7)}}{\kappa_1^{(1)} u_1^{(1)}} & \frac{\kappa_4 u_4^{(5)}}{u_1^{(6)}} & \frac{u_4^{(5)}}{u_1^{(1)}} \\ \frac{u_4^{(1)}}{\kappa_2 u_2^{(1)}} & \frac{\kappa_4 u_4^{(7)}}{u_2^{(7)}} & \frac{u_4^{(7)}}{u_2^{(1)}} \\ \frac{u_5^{(1)}}{\kappa_3 u_3^{(1)}} & \frac{\kappa_5 u_5^{(8)}}{u_3^{(8)}} & \frac{u_5^{(8)}}{u_3^{(1)}} \end{pmatrix} \qquad \Upsilon_{jk} = \begin{pmatrix} \frac{\kappa_1 w_1^{(1)}}{u_3^{(2)}} & \frac{\kappa_3 w_3^{(1)}}{w_2^{(1)}} & \frac{w_2^{(2)}}{w_3^{(1)}} \\ \frac{\kappa_1 w_1^{(2)}}{u_5^{(3)}} & \frac{\kappa_5 w_5^{(1)}}{w_4^{(1)}} & \frac{w_4^{(3)}}{w_5^{(1)}} \\ \frac{\kappa_2 w_2^{(2)}}{u_6^{(4)}} & \frac{\kappa_6 w_6^{(1)}}{w_4^{(3)}} & \frac{w_4^{(5)}}{w_6^{(1)}} \\ \frac{\kappa_3 w_3^{(2)}}{u_6^{(5)}} & \frac{\kappa_6 w_6^{(4)}}{w_5^{(3)}} & \frac{w_5^{(5)}}{w_6^{(4)}} \\ \frac{\kappa_1 w_1^{(6)}}{u_3^{(5)}} & \frac{\kappa_3 w_3^{(8)}}{w_2^{(7)}} & \frac{w_2^{(5)}}{w_3^{(8)}} \\ \frac{\kappa_1 w_1^{(1)}}{u_5^{(5)}} & \frac{\kappa_5 w_5^{(8)}}{w_4^{(7)}} & \frac{w_4^{(5)}}{w_5^{(8)}} \\ \frac{\kappa_2 w_2^{(1)}}{u_6^{(5)}} & \frac{\kappa_6 w_6^{(8)}}{w_4^{(1)}} & \frac{w_4^{(7)}}{w_6^{(8)}} \\ \frac{\kappa_3 w_3^{(1)}}{u_6^{(8)}} & \frac{\kappa_6 w_6^{(1)}}{w_5^{(1)}} & \frac{w_5^{(8)}}{w_6^{(1)}} \end{pmatrix}. \tag{59}$$

Here for any f it is implied that

$$f^{(2)} = \mathcal{R}_{1,2,3}^{(f)} \circ f^{(1)} \qquad f^{(3)} = \mathcal{R}_{1,2,3}^{(f)} \mathcal{R}_{1,4,5}^{(f)} \circ f^{(1)} \qquad f^{(4)} = \mathcal{R}_{1,2,3}^{(f)} \mathcal{R}_{1,4,5}^{(f)} \mathcal{R}_{2,5,6}^{(f)} \circ f^{(1)} \tag{60}$$

and

$$f^{(5)} = \mathcal{R}_{1,2,3}^{(f)} \mathcal{R}_{1,4,5}^{(f)} \mathcal{R}_{2,4,6}^{(f)} \mathcal{R}_{3,5,6}^{(f)} \circ f^{(1)}. \tag{61}$$

For the right-hand side of (44) we define for any f

$$f^{(8)} = \mathcal{R}_{3,5,6}^{(f)} \circ f^{(1)} \qquad f^{(7)} = \mathcal{R}_{3,5,6}^{(f)} \mathcal{R}_{2,4,6}^{(f)} \circ f^{(1)} \qquad f^{(6)} = \mathcal{R}_{3,5,6}^{(f)} \mathcal{R}_{2,4,6}^{(f)} \mathcal{R}_{1,4,5}^{(f)} \circ f^{(1)}. \tag{62}$$

Due to the validity of the functional tetrahedron equation the four times transformed function $f^{(\bar{5})} = \mathcal{R}_{3,5,6}^{(f)} \mathcal{R}_{2,4,6}^{(f)} \mathcal{R}_{1,4,5}^{(f)} \mathcal{R}_{1,2,3}^{(f)} \circ f^{(1)}$ coincides with (61).

Observe that, e.g.

$$\begin{array}{llll} u_1^{(3)} = u_1^{(4)} = u_1^{(5)} & u_2^{(2)} = u_2^{(3)} & u_2^{(4)} = u_2^{(5)} & u_3^{(2)} = u_3^{(3)} = u_3^{(4)} \\ u_4^{(8)} = u_4^{(1)} = u_4^{(2)} & u_4^{(4)} = u_4^{(5)} = u_4^{(6)} & u_5^{(1)} = u_5^{(2)} & u_5^{(3)} = u_5^{(4)} \\ u_5^{(5)} = u_5^{(6)} & u_6^{(1)} = u_6^{(2)} = u_6^{(3)}. & & \end{array} \tag{63}$$

The MTEs leave the following four 'centres' invariant, i.e. for $j = 1, \ldots, 8$ we have

$$\mathfrak{C}_1 = u_4^{(j)} u_5^{(j)} u_6^{(j)} \qquad \mathfrak{C}_2 = \frac{u_2^{(j)} u_3^{(j)}}{w_6^{(j)}} \qquad \mathfrak{C}_3 = \frac{w_3^{(j)} w_5^{(j)}}{u_1^{(j)}} \qquad \mathfrak{C}_4 = w_1^{(j)} w_2^{(j)} w_4^{(j)}. \tag{64}$$

For example, $\mathfrak{C}_1' = \mathfrak{C}_1$ because variables with indices 4, 5, 6 are not transformed by $\mathcal{R}_{1,2,3}^{(f)}$, then $\mathfrak{C}_1'' = \mathfrak{C}_1$ since $u_4 u_5$ is the centre of $\mathcal{R}_{1,4,5}^{(f)}$ and 6 does not appear in $\mathcal{R}_{1,4,5}^{(f)}$. $u_5 u_6$ is the centre of $\mathcal{R}_{2,5,6}^{(f)}$, etc.

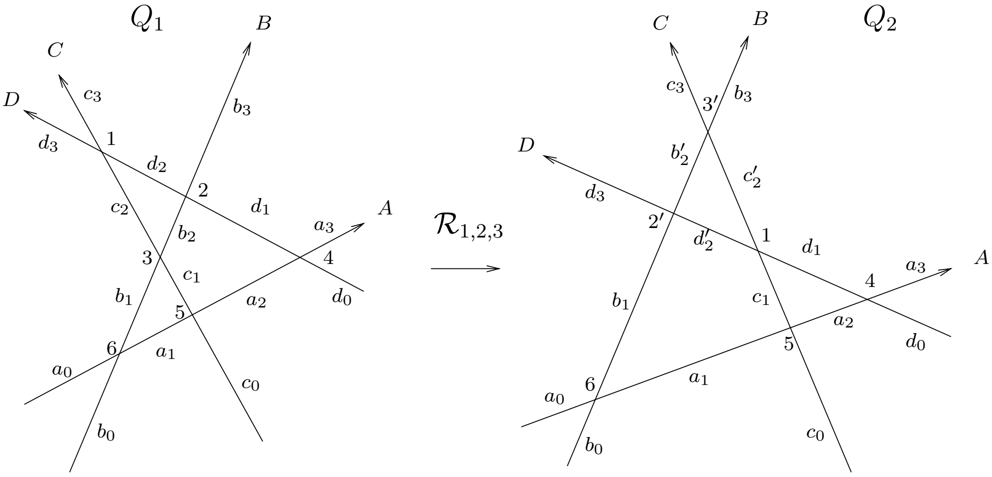

Figure 3. Parametrization of the arguments $u_i^{(j)}$, $w_i^{(j)}$ of the rational mapping $\mathcal{R}^{(j)}_{i,j,k}$ in terms of line-section ratios.

Equations (58) and (59) provide a 'free parametrization of the MTE' which generalizes the free parametrization of the single $\mathcal{R}_{1,2,3}$ introduced in remark 2. It corresponds to the following scenario: we start with 24 arbitrary complex numbers $u_j^{(1)}$, $w_j^{(1)}$, κ_j, $j = 1, \ldots, 8$, and apply repeatedly the functional mappings $\mathcal{R}^{(f)}$, choosing appropriately the phases of $u_j^{(k)}$, $w_j^{(k)}$. So we obtain a parametrization of the eight **R** matrices obeying the MTE.

The natural question arises: how many *independent* parameters has the MTE as a matrix identity? As we see already from the existence of the centres (64), some of the 24 parameters will occur only in certain combinations.

In order to determine the independent variables in a systematic way, in the next subsection we shall use a simple constructive procedure: we express the u_j and w_j in terms of line section ratios, the parametrization being designed so as to automatically conserve the centres of the mapping. This is in the same spirit as the introdution of τ-functions in the theory of solitons. We shall find that as a matrix identity the MTE may be parametrized by eight independent continuous parameters and eight discrete phases common for the left- and right-hand sides of MTE, and besides, each of the left- and right-hand sides contains four extra independent discrete parameters. In particular, the couplings κ_j can all be absorbed by a rescaling.

Note that one great advantage of the MTEs, which is not shared by Yang–Baxter equations, is that many different parametrizations can be found and can be chosen according to the particular calculations and applications one likes to do.

3.1. Parametrization in terms of line section ratios

We now parametrize the $u_i^{(j)}$, $w_i^{(j)}$ in terms of ratios of parameters which can be read off from the four line graphs ('quadrilaterals') shown in figure 2. Figure 3 gives an enlarged detail of figure 2 with labels attached, which are explained in the following.

In the quadrilateral Q_1 there are four directed lines A, B, C, D. The six vertex points cut the lines into four sections each. We denote the sections of line A by $a_0, a_1, \ldots, a_3$ (the indices increasing in the direction of the line). Analogously, we label the sections of lines B, C, D by $b_0, \ldots, d_3$. This way for each quadrilateral Q_j we have defined 16 variables.

Now, with each vertex i of each quadrilateral Q_j we associate a pair of variables $u_i^{(j)}$, $w_i^{(j)}$ (which determine the Fermat points in (58)), and we require these to be given in terms of the $a_0, a_1, \ldots, d_2, d_3$ of the respective quadrilateral as follows.

For a $u_i^{(j)}$, take Q_j and look from point i in the direction of the arrows and select the right-pointing of the two lines. Now $u_i^{(j)}$ is the ratio of the variable attached to the section *before* to the variable *after* the vertex. For the $w_i^{(j)}$ take the left-pointing line, and divide the variable after the vertex by the variable before the vertex. So, in the expressions for the $u_i^{(j)}$ the index of the numerator is one smaller than that for the denominator, the inverse is true for the $w_i^{(j)}$.

Passing from one quadrilateral to the next one, corresponding to a mapping $\mathcal{R}$, always three of the 'internal' lines are changed, and for distinction, we attach to these changed variables primes and daggers. The eight 'external' variables $a_0, a_3, \ldots, d_0, d_3$ are never changed by our mappings. Of the eight 'internal' variables (these have the indices 1 and 2) five are unchanged in each mapping. To collect these definitions, we define

$$U_j = \left[u_1^{(j)}, u_2^{(j)}, \ldots, u_5^{(j)}, u_6^{(j)}, w_1^{(j)}, w_2^{(j)}, \ldots, w_5^{(j)}, w_6^{(j)}\right]. \tag{65}$$

Then from figures 2 and 3 we read off, applying successively the anticlockwise mappings:

$$\begin{aligned}
U_1 &= \left[\frac{c_2}{c_3}, \frac{b_2}{b_3}, \frac{b_1}{b_2}, \frac{a_2}{a_3}, \frac{a_1}{a_2}, \frac{a_0}{a_1}, \frac{d_3}{d_2}, \frac{d_2}{d_1}, \frac{c_2}{c_1}, \frac{d_1}{d_0}, \frac{c_1}{c_0}, \frac{b_1}{b_0}\right] \\
U_2 &= \left[\frac{c_1}{c_2'}, \frac{b_1}{b_2'}, \frac{b_2'}{b_3}, \frac{a_2}{a_3}, \frac{a_1}{a_2}, \frac{a_0}{a_1}, \frac{d_2'}{d_1}, \frac{d_3}{d_2'}, \frac{c_3}{c_2'}, \frac{d_1}{d_0}, \frac{c_1}{c_0}, \frac{b_1}{b_0}\right] \\
U_3 &= \left[\frac{c_0}{c_1''}, \frac{b_1}{b_2'}, \frac{b_2'}{b_3}, \frac{a_1}{a_2''}, \frac{a_2''}{a_3}, \frac{a_0}{a_1}, \frac{d_1''}{d_0}, \frac{d_3}{d_2'}, \frac{c_3}{c_2'}, \frac{d_2'}{d_1''}, \frac{c_2'}{c_1''}, \frac{b_1}{b_0}\right] \\
U_4 &= \left[\frac{c_0}{c_1''}, \frac{b_0}{b_1'''}, \frac{b_2'}{b_3}, \frac{a_0}{a_1'''}, \frac{a_2''}{a_3}, \frac{a_1'''}{a_2''}, \frac{d_1''}{d_0}, \frac{d_2'''}{d_1''}, \frac{c_3}{c_2'}, \frac{d_3}{d_2'''}, \frac{c_2'}{c_1''}, \frac{b_2'}{b_1'''}\right] \\
U_5 &= \left[\frac{c_0}{c_1''}, \frac{b_0}{b_1'''}, \frac{b_1'''}{b_2^T}, \frac{a_0}{a_1'''}, \frac{a_1'''}{a_2^T}, \frac{a_2^T}{a_3}, \frac{d_1''}{d_0}, \frac{d_2'''}{d_1''}, \frac{c_2^T}{c_1''}, \frac{d_3}{d_2'''}, \frac{c_3}{c_2^T}, \frac{b_3}{b_2^T}\right].
\end{aligned} \tag{66}$$

We write a_2^T instead of a_2'''' and d_1^t instead of $d_1^{\dagger\dagger\dagger\dagger}$, etc. Transforming clockwise in figure 2, we get

$$\begin{aligned}
U_8 &= \left[\frac{c_2}{c_3}, \frac{b_2}{b_3}, \frac{b_0}{b_1^\dagger}, \frac{a_2}{a_3}, \frac{a_0}{a_1^\dagger}, \frac{a_1^\dagger}{a_2}, \frac{d_3}{d_2}, \frac{d_2}{d_1}, \frac{c_1^\dagger}{c_0}, \frac{d_1}{d_0}, \frac{c_2}{c_1^\dagger}, \frac{b_2}{b_1^\dagger}\right] \\
U_7 &= \left[\frac{c_2}{c_3}, \frac{b_1^\dagger}{b_2^{\dagger\dagger}}, \frac{b_0}{b_1^\dagger}, \frac{a_1}{a_2^{\dagger\dagger}}, \frac{a_0}{a_1^\dagger}, \frac{a_2^{\dagger\dagger}}{a_3}, \frac{d_3}{d_2}, \frac{d_1^{\dagger\dagger}}{d_0}, \frac{c_1^\dagger}{c_0}, \frac{d_2}{d_1^{\dagger\dagger}}, \frac{c_2}{c_1^\dagger}, \frac{b_3}{b_2^{\dagger\dagger}}\right]
\end{aligned}$$

etc. Transforming clockwise twice more, we arrive at an alternative expression for U_5, which for distinction we call $U_{\bar{5}}$:

$$U_{\bar{5}} = \left[\frac{c_0}{c_1^t}, \frac{b_0}{b_1^t}, \frac{b_1^t}{b_2^{\dagger\dagger}}, \frac{a_0}{a_1^{\dagger\dagger\dagger}}, \frac{a_1^{\dagger\dagger\dagger}}{a_2^{\dagger\dagger}}, \frac{a_2^{\dagger\dagger}}{a_3}, \frac{d_1^t}{d_0}, \frac{d_2^{\dagger\dagger\dagger}}{d_1^t}, \frac{c_2^{\dagger\dagger\dagger}}{c_1^t}, \frac{d_3}{d_2^{\dagger\dagger\dagger}}, \frac{c_3}{c_2^{\dagger\dagger\dagger}}, \frac{b_3}{b_2^{\dagger\dagger}}\right]. \tag{67}$$

We see that this particular parametrization in terms of ratios automatically incorporates the invariance of the centres of the mappings. For example, for $\mathcal{R}^{(1)}$ (in our notation (65) $u_j^{(2)}$ is the u_j' of (17)):

$$u_2^{(2)} u_3^{(2)} = u_2^{(1)} u_3^{(1)} \qquad u_1^{(2)} / w_3^{(2)} = u_1^{(1)} / w_3^{(1)} \qquad w_1^{(2)} w_2^{(2)} = w_1^{(1)} w_2^{(1)}.$$

Also more complicated conditions are fulfilled automatically, e.g.

$$u_1^{(4)} w_3^{(1)} w_5^{(1)} = u_1^{(1)} w_3^{(2)} w_5^{(3)}.$$

Observe that because in the mapping from Q_1 to Q_2 the line A keeps the same intersection points, there is no a_i' in the formulae for the U_j, similarly, there are no b_i'', c_i''', d_i^T, $d_i^\dagger$, $c_i^{\dagger\dagger}$, $b_i^{\dagger\dagger\dagger}$, a_i^t. So, altogether, there are 24 primed or daggered variables appearing. However, some coincide as we will see. From (13) and (14) we easily obtain the following relations, which allow their Nth powers to be expressed recursively in terms of the Nth powers of the unprimed variables:

$$\begin{aligned}
&b_2'^N c_2^N d_2^N = b_1^N c_3^N d_2^N + b_2^N c_3^N d_3^N + \kappa_1^N b_3^N c_2^N d_3^N \\
&\kappa_2^N b_2^N c_2'^N d_2^N = \kappa_1^N b_3^N c_1^N d_2^N + \kappa_3^N b_2^N c_3^N d_1^N + \kappa_1^N \kappa_3^N b_3^N c_2^N d_1^N \\
&b_2^N c_2^N d_2'^N = b_2^N c_1^N d_3^N + b_1^N c_1^N d_2^N + \kappa_3^N b_1^N c_2^N d_1^N \\
&a_2''^N c_1^N d_1^N = a_1^N c_2'^N d_1^N + a_2^N c_2'^N d_2'^N + \kappa_1^N a_3^N c_1^N d_2'^N \\
&\kappa_4^N a_2^N c_1''^N d_1^N = \kappa_1^N a_3^N c_0^N d_1^N + \kappa_5^N a_2^N c_2'^N d_0^N + \kappa_1^N \kappa_5^N a_3^N c_1^N d_0^N \\
&a_2^N c_1^N d_1''^N = a_2^N c_0^N d_2'^N + a_1^N c_0^N d_1^N + \kappa_5^N a_1^N c_1^N d_0^N \\
&a_1'''^N b_1^N d_2'^N = a_0^N b_2'^N d_2'^N + d_3^N a_1^N b_2'^N + \kappa_2^N d_3^N b_1^N a_2''^N \\
&\kappa_4^N a_1^N b_1'''^N d_2'^N = \kappa_2^N b_0^N a_2''^N d_2'^N + \kappa_6^N a_1^N b_2'^N d_1''^N + \kappa_2^N \kappa_6^N a_2''^N b_1^N d_1''^N \\
&a_1^N b_1^N d_2'''^N = b_0^N d_3^N a_1^N + a_0^N b_0^N d_2'^N + \kappa_6^N a_0^N b_1^N d_1''^N \\
&a_2^{T\,N} b_2'^N c_2'^N = b_3^N a_1'''^N c_2'^N + b_3^N c_3^N a_2''^N + \kappa_3^N a_3^N c_3^N b_2'^N \\
&\kappa_5^N a_2''^N b_2^{T\,N} c_2'^N = \kappa_3^N a_3^N b_1'''^N c_2'^N + \kappa_6^N b_3^N a_2''^N c_1''^N + \kappa_3^N \kappa_6^N a_3^N b_2'^N c_1''^N \\
&a_2''^N b_2'^N c_2^T = c_3^N a_2''^N b_1'''^N + a_1'''^N b_1'''^N c_2'^N + \kappa_6^N a_1'''^N b_2'^N c_1''^N \\
&a_1^{\dagger N} b_1^N c_1^N = a_0^N b_2^N c_1^N + a_1^N b_2^N c_2^N + \kappa_3^N a_2^N b_1^N c_2^N \\
&\kappa_5^N a_1^N b_1^{\dagger N} c_1^N = \kappa_3^N a_2^N b_0^N c_1^N + \kappa_6^N a_1^N b_2^N c_0^N + \kappa_3^N \kappa_6^N a_2^N b_1^N c_0^N \\
&a_1^N b_1^N c_1^{\dagger N} = a_1^N b_0^N c_2^N + a_0^N b_0^N c_1^N + \kappa_6^N a_0^N b_1^N c_0^N \\
&a_2^N b_2^N d_1^{\dagger\dagger N} = a_2^N b_1^{\dagger N} d_2^N + a_1^{\dagger N} b_1^{\dagger N} d_1^N + \kappa_6^N a_1^{\dagger N} b_2^N d_0^N \\
&a_2^{\dagger\dagger N} b_2^N d_1^N = b_3^N a_1^{\dagger N} d_1^N + b_3^N a_2^N d_2^N + \kappa_2^N a_3^N b_2^N d_2^N \\
&\kappa_4^N a_2^N b_2^{\dagger\dagger N} d_1^N = \kappa_2^N a_3^N b_1^{\dagger N} d_1^N + \kappa_6^N b_3^N a_2^N d_0^N + \kappa_2^N \kappa_6^N a_3^N b_2^N d_0^N \\
&a_1^{\dagger\dagger\dagger N} c_2^N d_2^N = a_0^N c_3^N d_2^N + c_3^N d_3^N a_1^{\dagger N} + \kappa_1^N d_3^N a_2^{\dagger\dagger N} c_2^N \\
&\kappa_4^N a_1^{\dagger N} c_2^{\dagger\dagger\dagger N} d_2^N = \kappa_1^N a_2^{\dagger\dagger} c_1^{\dagger N} d_2^N + \kappa_5^N c_3^N a_1^{\dagger N} d_1^{\dagger\dagger N} + \kappa_1^N \kappa_5^N a_2^{\dagger\dagger N} c_2^N d_1^{\dagger\dagger N} \\
&a_1^{\dagger N} c_2^N d_2^{\dagger\dagger\dagger N} = d_3^N a_1^{\dagger N} c_1^{\dagger N} + a_0^N c_1^{\dagger N} d_2^N + \kappa_5^N a_0^N c_2^N d_1^{\dagger\dagger N} \\
&b_1^{t\,N} c_1^{\dagger N} d_1^{\dagger\dagger N} = b_0^N c_2^{\dagger\dagger\dagger N} d_1^{\dagger\dagger N} + b_1^{\dagger N} c_2^{\dagger\dagger\dagger N} d_2^{\dagger\dagger\dagger N} + \kappa_1^N b_2^{\dagger\dagger N} c_1^{\dagger N} d_2^{\dagger\dagger\dagger N} \\
&\kappa_2^N b_1^{\dagger N} c_1^{t\,N} d_1^{\dagger\dagger N} = \kappa_1^N c_0^N b_2^{\dagger\dagger N} d_1^{\dagger\dagger N} + \kappa_3^N d_0^N b_1^{\dagger N} c_2^{\dagger\dagger\dagger N} + \kappa_1^N \kappa_3^N d_0^N b_2^{\dagger\dagger N} c_1^{\dagger N} \\
&b_1^{\dagger N} c_1^{\dagger N} d_1^{t\,N} = c_0^N b_1^{\dagger N} d_2^{\dagger\dagger\dagger N} + b_0^N c_0^N d_1^{\dagger\dagger N} + \kappa_3^N b_0^N d_0^N c_1^{\dagger N}.
\end{aligned} \tag{68}$$

Using these relations one verifies by straightforward substitution in (66) and (67) that

$$U_5 = U_{\bar{5}} \tag{69}$$

i.e. that the functional tetrahedron equations are satisfied. Equation (69) tells us that there are eight direct relations between the primed and daggered variables, e.g. $c_1'' = c_1^t$, etc so that in fact due to the functional tetrahedron equations the last eight equations of (68) are superfluous. This will be important for the discussion of the freedom of phase choices when taking Nth roots.

Written in terms of our parameters a_i, b_i, etc the arguments of the $\mathcal{R}^{(j)}$ $(j = 1, \dots, 8)$, see (58), are

$$\begin{aligned} &x_1^{(j)} = \frac{1}{\sqrt{\omega}}\mathcal{X}_{j1} \qquad x_2^{(j)} = \frac{1}{\sqrt{\omega}}\mathcal{X}_{j2} \qquad x_3^{(j)} = \frac{1}{\omega}\mathcal{X}_{j3} \qquad x_4^{(j)} = \frac{x_1^{(j)}x_2^{(j)}}{\omega x_3^{(j)}} \\ &y_{31}^{(j)} = \mathcal{Y}_{j1} \qquad y_{41}^{(j)} = \frac{1}{\sqrt{\omega}}\mathcal{Y}_{j2} \qquad y_{32}^{(j)} = \mathcal{Y}_{j3} \end{aligned} \tag{70}$$

where (again we do not write out $x_4^{(j)}$)

$$\mathcal{X}_{jk} = \begin{pmatrix} \frac{b_2c_3}{\kappa_1 b_3 c_2} & \frac{\kappa_2 b_1 c_2'}{b_2' c_1} & \frac{b_1 c_3}{b_2' c_2} \\ \frac{a_2 c_2'}{\kappa_1 a_3 c_1} & \frac{\kappa_4 a_1 c_1''}{a_2'' c_0} & \frac{a_1 c_2'}{a_2'' c_1} \\ \frac{a_1 b_2'}{\kappa_2 a_2'' b_1} & \frac{\kappa_4 a_0 b_1'''}{b_0 a_1'''} & \frac{a_0 b_2'}{a_1''' b_1} \\ \frac{a_2'' b_3}{\kappa_3 b_2' a_3} & \frac{\kappa_5 a_1''' b_2^T}{b_1''' a_2^T} & \frac{a_1''' b_3}{a_2^T b_2'} \\ \frac{b_1^\dagger c_2^{\dagger\dagger\dagger}}{\kappa_1 b_2^{\dagger\dagger} c_1^\dagger} & \frac{\kappa_2 b_0 c_1^t}{b_1^t c_0} & \frac{b_0 c_2^{\dagger\dagger\dagger}}{b_1^t c_1^\dagger} \\ \frac{a_1^\dagger c_3}{\kappa_1 a_2^{\dagger\dagger} c_2} & \frac{\kappa_4 a_0 c_2^{\dagger\dagger\dagger}}{c_1^\dagger a_1^{\dagger\dagger\dagger}} & \frac{a_0 c_3}{a_1^{\dagger\dagger\dagger} c_2} \\ \frac{a_2 b_3}{\kappa_2 a_3 b_2} & \frac{\kappa_4 a_1^\dagger b_2^{\dagger\dagger}}{a_2^{\dagger\dagger} b_1^\dagger} & \frac{a_1^\dagger b_3}{a_2^{\dagger\dagger} b_2} \\ \frac{a_1 b_2}{\kappa_3 a_2 b_1} & \frac{\kappa_5 a_0 b_1^\dagger}{b_0 a_1^\dagger} & \frac{a_0 b_2}{a_1^\dagger b_1} \end{pmatrix} \qquad \mathcal{Y}_{jk} = \begin{pmatrix} \frac{\kappa_1 d_3 b_3}{d_2 b_2'} & \frac{\kappa_3 c_2 d_1}{c_1 d_2} & \frac{d_3 c_1}{d_2' c_2} \\ \frac{\kappa_1 d_2' a_3}{d_1 a_2''} & \frac{\kappa_5 c_1 d_0}{c_0 d_1} & \frac{d_2' c_0}{d_1'' c_1} \\ \frac{\kappa_2 d_3 a_2''}{d_2' a_1'''} & \frac{\kappa_6 b_1 d_1''}{b_0 d_2'} & \frac{d_3 b_0}{d_2''' b_1} \\ \frac{\kappa_3 c_3 a_3}{c_2' a_2^T} & \frac{\kappa_6 b_2' c_1''}{b_1''' c_2'} & \frac{c_3 b_1'''}{c_2^T b_2'} \\ \frac{\kappa_1 d_2^{\dagger\dagger\dagger} b_2^{\dagger\dagger}}{d_1^{\dagger\dagger} b_1^t} & \frac{\kappa_3 c_1^\dagger d_0}{c_0 d_1^{\dagger\dagger}} & \frac{d_2^{\dagger\dagger\dagger} c_0}{d_1^t c_1^\dagger} \\ \frac{\kappa_1 d_3 a_2^{\dagger\dagger}}{d_2 a_1^{\dagger\dagger\dagger}} & \frac{\kappa_5 c_2 d_1^{\dagger\dagger}}{c_1^\dagger d_2} & \frac{d_3 c_1^\dagger}{d_2^{\dagger\dagger\dagger} c_2} \\ \frac{\kappa_2 d_2 a_3}{d_1 a_2^{\dagger\dagger}} & \frac{\kappa_6 b_2 d_0}{b_1^\dagger d_1} & \frac{d_2 b_1^\dagger}{d_1^{\dagger\dagger} b_2} \\ \frac{\kappa_3 c_2 a_2}{c_1 a_1^\dagger} & \frac{\kappa_6 b_1 c_0}{b_0 c_1} & \frac{c_2 b_0}{c_1^\dagger b_1} \end{pmatrix}. \tag{71}$$

To check the validity of all the Fermat relations ${x_i^{(j)}}^N + {y_i^{(j)}}^N = 1$ requires using the transformation equations (68). The relations (68) involve the Nth powers of the variables, so to use them to get the Fermat coordinates, we have to take Nth roots, which entails discrete phase choices. The centres (64) are just ratios of the external variables:

$$\mathfrak{C}_1 = \frac{a_0}{a_3} \qquad \mathfrak{C}_2 = \frac{b_0}{b_3} \qquad \mathfrak{C}_3 = \frac{c_3}{c_0} \qquad \mathfrak{C}_4 = \frac{d_3}{d_0}. \tag{72}$$

The external variables $a_0, a_3, \dots, d_0, d_3$ are irrelevant and serve mainly to express all quantities in terms of ratios. We may choose them simply all to be unity. The 'coupling constants' κ_j may all be eliminated by rescaling the eight relevant variables $a_1, a_2, \dots, d_1, d_2$ as follows:

$$\begin{aligned} &a_1 = \frac{\kappa_1\kappa_2}{\kappa_5}\,\overline{a_1} \qquad a_2 = \frac{\kappa_1\kappa_6}{\kappa_3}\,\overline{a_2} \qquad b_1 = \frac{\kappa_1}{\kappa_3\kappa_5}\,\overline{b_1} \qquad b_2 = \frac{\kappa_1\kappa_6}{\kappa_2\kappa_3}\,\overline{b_2} \\ &c_1 = \frac{\kappa_1\kappa_6}{\kappa_3\kappa_5}\,\overline{c_1} \qquad c_2 = \frac{\kappa_6}{\kappa_2\kappa_3}\,\overline{c_2} \qquad d_1 = \frac{\kappa_1\kappa_6}{\kappa_3}\,\overline{d_1} \qquad d_2 = \frac{\kappa_5\kappa_6}{\kappa_2}\,\overline{d_2}. \end{aligned} \tag{73}$$

This entails a corresponding rescaling of the primed and daggered variables, e.g.

$$
\begin{array}{llll}
b_2' = \dfrac{\kappa_1\kappa_2}{\kappa_5\kappa_6}\,\overline{b_1'} & c_2' = \dfrac{\kappa_1}{\kappa_5}\,\overline{c_2'} & d_2' = \dfrac{\kappa_1\kappa_2}{\kappa_5}\,\overline{d_2'} & a_2'' = \dfrac{\kappa_1\kappa_2\kappa_3}{\kappa_5\kappa_6}\,\overline{a_2''} \\
c_1'' = \dfrac{\kappa_3}{\kappa_4\kappa_6}\,\overline{c_1''} & d_1'' = \dfrac{\kappa_2\kappa_3}{\kappa_6}\,\overline{d_1''} & a_1''' = \dfrac{\kappa_2\kappa_3}{\kappa_6}\,\overline{a_1'''} & b_1''' = \dfrac{\kappa_2\kappa_3}{\kappa_4\kappa_6}\,\overline{b_1'''} \\
d_2''' = \dfrac{\kappa_3\kappa_5}{\kappa_1}\,\overline{d_2'''} & a_2^T = \dfrac{\kappa_3\kappa_5}{\kappa_1}\,\overline{a_2^T} & b_2^T = \dfrac{\kappa_3}{\kappa_1\kappa_4}\,\overline{b_2^T} & c_2^T = \dfrac{\kappa_3\kappa_5}{\kappa_1\kappa_4}\,\overline{c_2^T} \\
a_1^\dagger = \dfrac{\kappa_5\kappa_6}{\kappa_2}\,\overline{a_1^\dagger} & b_1^\dagger = \dfrac{\kappa_6}{\kappa_2}\,\overline{b_1^\dagger} & c_1^\dagger = \dfrac{\kappa_5\kappa_6}{\kappa_1\kappa_2}\,\overline{c_1^\dagger} & \ldots .
\end{array}
\tag{74}
$$

So we can simplify equations (68) and (71) by taking $a_0 = a_3 = b_0 = \cdots = d_3 = 1$ and $\kappa_1 = \kappa_2 = \cdots = \kappa_6 = 1$ and replacing the relevant variables by their overlined counterparts.

3.2. The choice of discrete phases

As we have already mentioned, apart from the eight continuous parameters just discussed, the left-hand and right-hand sides of the MTE depend on phase choices arising from taking Nth roots. We investigate how many independent choices can be made and whether these affect the MTEs.

From (69) and (66), (67) we see that the left-hand side (LHS) and right-hand side (RHS) of the MTE have eight arbitrary *common* phases of

$$
\begin{array}{llll}
a_2^T = a_2^{\dagger\dagger} & b_2^T = b_2^{\dagger\dagger} & c_2^T = c_2^{\dagger\dagger\dagger} & d_2''' = d_2^{\dagger\dagger\dagger} \\
a_1''' = a_1^{\dagger\dagger\dagger} & b_1''' = b_1^t & c_1'' = c_1^t & d_1'' = d_1^t.
\end{array}
$$

These phases correspond to the phases of $u_1^{(5)}$ etc. Furthermore, the LHS of the MTE contains four internal phases of

$$c_2',\, b_2',\, d_2',\, a_2''$$

while the RHS contains four internal phases of

$$a_1^\dagger,\, c_1^\dagger,\, b_1^\dagger,\, d_1^{\dagger\dagger}.$$

In which way does the LHS depend on its internal phases? Consider, e.g., the shift

$$c_2' \mapsto q^{-1} c_2'.$$

According to (71) this shift changes the following Fermat coordinates:

$$
\begin{array}{lll}
x_2^{(1)} \mapsto q^{-1}x_2^{(1)} & x_4^{(1)} \mapsto q^{-1}x_4^{(1)} & \\
x_1^{(2)} \mapsto q^{-1}x_1^{(2)} & x_3^{(2)} \mapsto q^{-1}x_3^{(2)} & y_1^{(4)} \mapsto q y_1^{(4)}.
\end{array}
$$

Now note that for $q = \omega$

$$W_{(q^{-1}x,y)}(n) = \frac{y}{1-x} W_{(x,y)}(n-1) \qquad W_{(x,qy)}(n) = q^n W_{(x,y)}(n).$$

Therefore, our shift produces

$$
\begin{aligned}
\langle i_1, i_2, i_3 | R^{(1)} | j_1, j_2, j_3\rangle &\mapsto \frac{y_2^{(1)}}{y_4^{(1)}} \frac{1 - x_4^{(1)}}{1 - x_2^{(1)}} \langle i_1, i_2, i_3 | R^{(1)} | j_1 + 1, j_2, j_3\rangle\, q^{-j_3} \\
\langle j_1, i_4, i_5 | R^{(2)} | k_1, j_4, j_5\rangle &\mapsto \frac{y_1^{(2)}}{y_3^{(2)}} \frac{1 - x_3^{(2)}}{1 - x_1^{(2)}} \langle j_1 + 1, i_4, i_5 | R^{(2)} | k_1, j_4, j_5\rangle\, q^{j_5} \\
\langle j_3, j_5, j_6 | R^{(4)} | k_3, k_5, k_6\rangle &\mapsto q^{j_3 - j_5} \langle j_3, j_5, j_6 | R^{(4)} | k_3, k_5, k_6\rangle.
\end{aligned}
$$

We see that the change considered produces a simple scalar factor and does not change the matrix structure of the LHS of the MTE:

$$\mathrm{LHS}(q^{-1}c_2') = \frac{y_2^{(1)}}{y_4^{(1)}} \frac{1-x_4^{(1)}}{1-x_2^{(1)}} \frac{y_1^{(2)}}{y_3^{(2)}} \frac{1-x_3^{(2)}}{1-x_1^{(2)}} \,\mathrm{LHS}(c_2').$$

In this way we may convince ourselves:

- A change of the phases of any of the eight internal variables produces only extra scalar factors for the LHS or RHS of the MTE, while the external matrix structure does not change.
- A change of the phases for any of the eight external variables produces extra scalar factors both for the LHS and RHS of the MTE, and besides it produces a change of the external matrix structure: a shift of the indices $i_1, \ldots, k_6$ and some multipliers $q^{\pm i_1}, \ldots, q^{\pm k_6}$. However, this change is the same for the LHS and RHS of the MTE.

4. Explicit form of the MTE for $N = 2$

For $N = 2$, using combined indices $i = 1 + i_1 + 2i_2 + 4i_3$; $k = 1 + k_1 + 2k_2 + 4k_3$, we can give $(R)_{i1,i2,i3}^{k1,k2,k3}$ explicitly in a simple matrix form. We define

$$Y_k = \frac{y_k}{1+x_k} = \sqrt{\frac{1-x_k}{1+x_k}} \qquad \text{for} \quad k = 1, 2, 3, 4$$
$$Z_{ik} = \frac{Y_i}{Y_k} \qquad \text{for} \quad ik = 13, 14, 23, 24 \qquad Z_{12} = Y_1 Y_2 \qquad Z_{34} = \frac{1}{Y_3 Y_4} \tag{75}$$

and get

$$\mathbf{R}_i^k = \begin{pmatrix} 1 & Z_{24} & 0 & 0 & 0 & 0 & Z_{23} & -Z_{34} \\ Z_{13} & Z_{13}Z_{24} & 0 & 0 & 0 & 0 & -Z_{12} & Z_{14} \\ 0 & 0 & Z_{13}Z_{24} & Z_{13} & Z_{14} & -Z_{12} & 0 & 0 \\ 0 & 0 & Z_{24} & 1 & -Z_{34} & Z_{23} & 0 & 0 \\ 0 & 0 & Z_{23} & Z_{34} & 1 & -Z_{24} & 0 & 0 \\ 0 & 0 & Z_{12} & Z_{14} & -Z_{13} & Z_{13}Z_{24} & 0 & 0 \\ Z_{14} & Z_{12} & 0 & 0 & 0 & 0 & Z_{13}Z_{24} & -Z_{13} \\ Z_{34} & Z_{23} & 0 & 0 & 0 & 0 & -Z_{24} & 1 \end{pmatrix}_{ik}. \tag{76}$$

The determinant can be calculated directly:

$$\det \mathbf{R} = \left(Y_1 Y_2 \frac{\left(Y_3^2+1\right)\left(Y_4^2-1\right)}{Y_3^2\, Y_4^2} \right)^4. \tag{77}$$

For $N = 2$ we can also write the MTE quite explicitly. We shall write equation (44) as

$$\mathbf{\Theta}_{i_1,i_2,i_3,i_4,i_5,i_6}^{k_1,k_2,k_3,k_4,k_5,k_6} = \rho \overline{\mathbf{\Theta}}_{i_1,i_2,i_3,i_4,i_5,i_6}^{k_1,k_2,k_3,k_4,k_5,k_6} \tag{78}$$

where in obvious correspondence $\mathbf{\Theta}$ and $\overline{\mathbf{\Theta}}$ are defined to be the left- and right-hand sums of the products of four R matrices. We use (75) and abbreviate $Y_i^{(j)}$ by Y_{ij}, where i labels the four points on the Fermat curve $x_1, x_2, x_3, x_4 = x_1 x_2/(\omega x_3)$, and $j = 1, \ldots, 8$ denote the eight arguments of the $\mathcal{R}^{(j)}$. The left-hand side of (78) is, using (33):

$$\begin{aligned}\Theta^{k_1,k_2,k_3,k_4,k_5,k_6}_{i_1,i_2,i_3,i_4,i_5,i_6} &= (-1)^{k_3k_6}\frac{Y_{11}^{i_2+i_1}Y_{23}^{k_4+k_2}Y_{24}^{k_5+k_3}}{Y_{42}^{i_4+k_1}}\\ &\times\sum_{j_1,j_2,j_3,j_4,j_5,j_6}\delta_{i_2+i_3,\,j_2+j_3}\delta_{i_4+i_5,\,j_4+j_5}\delta_{j_4+i_6,\,k_4+j_6}\delta_{j_5+j_6,\,k_5+k_6}\\ &\times\frac{(-1)^{j_1(j_3-j_5)+j_6(k_2-j_2)-j_3(i_1+k_6)+k_1j_5}Y_{21}^{j_2+j_1}Y_{12}^{i_4+j_1}Y_{22}^{j_4+k_1}Y_{13}^{j_4+j_2}Y_{14}^{j_5+j_3}}{Y_{31}^{j_2+i_1}Y_{41}^{i_2+j_1}Y_{32}^{j_4+j_1}Y_{33}^{k_4+j_2}Y_{43}^{j_4+k_2}Y_{34}^{k_5+j_3}Y_{44}^{j_5+k_3}}\end{aligned}$$

where all exponents are understood modulo 2, i.e. being just 0 or 1. We can use three of the δ to eliminate the sums over, e.g., j_2, j_4, j_6. Then the last δ gives a compatibility condition with the result that if $i_4+i_5+i_6+k_4+k_5+k_6$ is odd, the component of Θ vanishes and these components of the MTE are trivial. These are half of the 2^{12} components. We also see that, if these occur at all, Y_{11}, Y_{23}, Y_{24} and Y_{42} will factorize (Y_{11} appears if i_1+i_2 is odd, etc). Each nonzero component has a sum over eight terms on each side. The result is

$$\begin{aligned}\Theta^{k_1,k_2,k_3,k_4,k_5,k_6}_{i_1,i_2,i_3,i_4,i_5,i_6} &= \delta_{i_4+i_5+i_6,k_4+k_5+k_6}(-1)^{\gamma}\frac{Y_{11}^{i_2+i_1}Y_{23}^{k_4+k_2}Y_{24}^{k_5+k_3}}{Y_{42}^{i_4+k_1}}\\ &\times\sum_{j_1,j_2,j_4}\frac{(-1)^{\beta}Y_{21}^{j_2+j_1}Y_{12}^{i_4+j_1}Y_{22}^{j_4+k_1}Y_{13}^{j_4+j_2}Y_{14}^{j_4+j_2+\iota}}{Y_{31}^{j_2+i_1}Y_{41}^{i_2+j_1}Y_{32}^{j_4+j_1}Y_{33}^{k_4+j_2}Y_{43}^{j_4+k_2}Y_{34}^{k_5+i_2+i_3+j_2}Y_{44}^{i_4+i_5+j_4+k_3}}\end{aligned} \tag{79}$$

where $\iota = i_2+i_3+i_4+i_5$ and

$$\begin{aligned}\beta &= j_1j_2+j_2j_4+j_4j_1+j_1\iota+j_2(i_1+i_6+k_4+k_6)+j_4(k_1+k_2)\\ \gamma &= k_1(i_4+i_5)+k_2(i_6+k_4)+(i_2+i_3)(i_1+k_6)+k_3k_6.\end{aligned}$$

The analogous expression for the right-hand side is

$$\begin{aligned}\overline{\Theta}^{k_1,k_2,k_3,k_4,k_5,k_6}_{i_1,i_2,i_3,i_4,i_5,i_6} &= \delta_{i_4+i_5+i_6,k_4+k_5+k_6}(-1)^{\overline{\gamma}}\frac{Y_{25}^{k_2+k_1}Y_{18}^{i_5+i_3}Y_{17}^{i_4+i_2}}{Y_{36}^{k_4+i_1}}\\ &\times\sum_{j_1,j_2,j_4}\frac{(-1)^{\overline{\beta}}Y_{16}^{j_4+i_1}Y_{15}^{j_2+j_1}Y_{26}^{j_1+k_4}Y_{27}^{j_4+j_2}Y_{28}^{j_4+j_2+\overline{\iota}}}{Y_{38}^{j_4+i_3+k_4+k_5}Y_{48}^{i_5+j_2+k_2+k_3}Y_{37}^{i_2+j_4}Y_{47}^{i_4+j_2}Y_{46}^{j_1+j_4}Y_{35}^{j_1+k_2}Y_{45}^{k_1+j_2}}\end{aligned} \tag{80}$$

with $\overline{\iota} = k_2+k_3+k_4+k_5$, and

$$\begin{aligned}\overline{\beta} &= j_1(k_3+k_5)+j_2(j_4+i_4)+j_4(k_2+k_3+i_3)\\ \overline{\gamma} &= i_1k_5+i_2k_6+k_1k_3+(i_4+k_6)(k_2+k_3+i_3)\end{aligned}$$

and again all exponents are understood mod 2.

We give two examples of non-trivial components of the MTEs: first we consider the component

$$\Theta^{0,0,0,0,0,0}_{0,0,0,0,0,0} = \rho\overline{\Theta}^{0,0,0,0,0,0}_{0,0,0,0,0,0}.$$

This is, written more explicitly, using the abbreviations of (75) adding the index j: $Z_{ik,j} = Y_i^{(j)}\big/Y_k^{(j)}$ for $ik = 13, 14, 23, 24$, $Z_{12,j} = Y_1^{(j)}Y_2^{(j)}$, $Z_{34,j} = 1\big/\big(Y_3^{(j)}Y_4^{(j)}\big)$:

$$\begin{aligned}1 &+ Z_{24,1}Z_{13,2}+(Z_{23,1}-Z_{34,1}Z_{13,2})Z_{13,3}Z_{13,4}+(Z_{23,2}-Z_{12,2}Z_{24,1})Z_{14,3}Z_{14,4}\\ &\quad -(Z_{23,1}Z_{23,2}+Z_{34,1}Z_{12,2})Z_{34,3}Z_{34,4}\\ &= \rho\{1+Z_{24,6}Z_{13,5}+(Z_{14,5}+Z_{24,6}Z_{34,5})Z_{24,8}Z_{24,7}\\ &\quad +(Z_{14,6}+Z_{12,6}Z_{13,5})Z_{23,8}Z_{23,7}-(Z_{14,6}Z_{14,5}+Z_{12,6}Z_{34,5})Z_{34,8}Z_{34,7}\}.\end{aligned} \tag{81}$$

Another component:

$$\mathbf{\Theta}^{0,0,1,1,0,0}_{0,0,1,0,1,0} = \rho\, \overline{\mathbf{\Theta}}^{0,0,1,1,0,0}_{0,0,1,0,1,0}$$

reads analogously

$$\begin{aligned}(1 + Z_{24,1}Z_{13,2})&Z_{23,3}Z_{23,4} - (Z_{23,1} - Z_{13,2}Z_{34,1})Z_{12,3}Z_{12,4}\\ &+ (Z_{23,2}-Z_{24,1}Z_{12,2})Z_{13,3}Z_{24,3}Z_{13,4}Z_{24,4} + (Z_{23,1}Z_{23,2} + Z_{34,1}Z_{12,2})Z_{24,3}Z_{24,4}\\ &= \rho\{Z_{23,6} - Z_{34,6}Z_{13,5} + (Z_{23,6}Z_{14,5} - Z_{34,6}Z_{34,5})Z_{24,8}Z_{24,7}\\ &+ (Z_{24,6} - Z_{13,5})Z_{13,6}Z_{23,8}Z_{23,7} + (Z_{34,5}Z_{13,6} - Z_{13,6}Z_{24,6}Z_{14,5})Z_{34,7}Z_{34,8}\}.\end{aligned} \tag{82}$$

Each equation appears *eight* times for different components. In order to write the symmetries compactly, we introduce the following three mappings a, b, c of the upper or lower indices:

$$\begin{aligned}a(i_1, i_2, i_3, i_4, i_5, i_6) &= (i_1 + 1, i_2 + 1, i_3, i_4 + 1, i_5, i_6)\\ b(i_1, i_2, i_3, i_4, i_5, i_6) &= (i_1, i_2, i_3 + 1, i_4, i_5 + 1, i_6)\\ c(i_1, i_2, i_3, i_4, i_5, i_6) &= (i_1, i_2, i_3, i_4, i_5, i_6 + 1)\end{aligned}$$

and

$$ab(i_1, i_2, i_3, i_4, i_5, i_6) = a(b(i_1, i_2, i_3, i_4, i_5, i_6)) \quad \text{etc} \tag{83}$$

the same also for the k_j instead of the i_j. Of course, the indices are always taken mod 2. Altogether, for $N = 2$ there are 2^8 different nontrivial components, an independent set is (the same for the $\overline{\mathbf{\Theta}}$)

$$\mathbf{\Theta}^{k_1,k_2,k_3,0,0,0}_{i_1,i_2,i_3,0,0,0}\,, \qquad \mathbf{\Theta}^{k_1,k_2,k_3,1,1,0}_{i_1,i_2,i_3,0,0,0}\,, \qquad \mathbf{\Theta}^{k_1,k_2,k_3,1,0,1}_{i_1,i_2,i_3,0,0,0}\,, \qquad \mathbf{\Theta}^{k_1,k_2,k_3,0,1,1}_{i_1,i_2,i_3,0,0,0}$$
$$\text{for} \quad i_1, i_2, i_3, k_1, k_2, k_3 = 0, 1. \tag{84}$$

Proposition 4. *For $N = 2$ the components of the left-hand side of the modified tetrahedron equation satisfy the following symmetry relations:*

$$\begin{aligned}\mathbf{\Theta}^{k_1,k_2,k_3,k_4,k_5,k_6}_{i_1,i_2,i_3,0,0,0} &= \mathbf{\Theta}^{a(k_1,k_2,k_3,k_4,k_5,k_6)}_{a(i_1,i_2,i_3,0,0,0)} = (-1)^{i_1+k_1}\mathbf{\Theta}^{b(k_1,k_2,k_3,k_4,k_5,k_6)}_{b(i_1,i_2,i_3,0,0,0)}\\ &= (-1)^{i_1+k_1}\mathbf{\Theta}^{ab(k_1,k_2,k_3,k_4,k_5,k_6)}_{ab(i_1,i_2,i_3,0,0,0)} = (-1)^{i_2+i_3+k_2+k_3}\mathbf{\Theta}^{c(k_1,k_2,k_3,k_4,k_5,k_6)}_{c(i_1,i_2,i_3,0,0,0)}\\ &= (-1)^{i_2+i_3+k_2+k_3}\mathbf{\Theta}^{ac(k_1,k_2,k_3,k_4,k_5,k_6)}_{ac(i_1,i_2,i_3,0,0,0)} = (-1)^{i_1+i_2+i_3+k_1+k_2+k_3}\mathbf{\Theta}^{bc(k_1,k_2,k_3,k_4,k_5,k_6)}_{bc(i_1,i_2,i_3,0,0,0)}\\ &= (-1)^{i_1+i_2+i_3+k_1+k_2+k_3}\mathbf{\Theta}^{abc(k_1,k_2,k_3,k_4,k_5,k_6)}_{abc(i_1,i_2,i_3,0,0,0)}\end{aligned} \tag{85}$$

where $i_1, i_2, i_3, k_1, k_2, \ldots, k_5, k_6 = 0, 1$. Here the mappings a, b, c are defined as in (83). The same equations are valid for the right-hand components, i.e. for $\mathbf{\Theta}$ replaced by $\overline{\mathbf{\Theta}}$. For $k_4 + k_5 + k_6$ odd, these equations are trivial.

Proof. Use the explicit formulae (79) and (80). For the relations involving the mapping a we shift the summation indices j_1, j_2, j_4. Then all exponents of the Y-factors are unchanged. No phase appears, since the shift in γ ($\overline{\gamma}$) is compensated by the shift in β ($\overline{\beta}$). For the relations involving the mappings b and/or c, also all exponents of the Y are unchanged. The phase factors appear from the shifts in γ or $\overline{\gamma}$. □

The terms

$$Z_{ik,j} = y^{(j)}_{ik}\frac{1 + x^{(j)}_k}{1 + x^{(j)}_i} \qquad ik = 13, 14, 23, 24$$

$$Z_{12,j} = \frac{y^{(j)}_{13}y^{(j)}_{23}\left(1 - x^{(j)2}_3\right)}{\left(1 + x^{(j)}_1\right)\left(1 + x^{(j)}_2\right)} \qquad Z_{34,j} = \frac{\left(1 + x^{(j)}_3\right)\left(1 + x^{(j)}_4\right)}{y^{(j)}_{31}y^{(j)}_{41}\left(1 - x^{(j)2}_1\right)}$$

which appear in (81) and (82) have the following explicit form, taking all boundary coefficients and all κ_j to be unity, compare (73) and (74):

$$
\begin{aligned}
Z_{13,1} &= \frac{(b_2'c_2 - b_1)d_2}{c_2 - \mathrm{i}b_2} & Z_{14,1} &= \frac{(b_2c_2' - c_1)d_2}{(b_2 + \mathrm{i}c_2)d_1} \\
Z_{23,1} &= \frac{(b_2'c_2 - b_1)d_2'}{c_1b_2' - \mathrm{i}b_1c_2'} & Z_{24,1} &= \frac{(b_2c_2' - c_1)d_2'}{(b_1c_2' + \mathrm{i}b_2'c_1)d_1} \\
& \dots \\
Z_{23,4} &= \frac{\left(a_2^{\dagger\dagger}b_2' - a_1^{\dagger\dagger\dagger}\right)c_2^{\dagger\dagger\dagger}}{b_1'''a_2^{\dagger\dagger} - \mathrm{i}a_1^{\dagger\dagger\dagger}b_2^{\dagger\dagger}}; & Z_{24,4} &= \frac{\left(a_2''b_2^{\dagger\dagger} - b_1'''\right)c_2^{\dagger\dagger\dagger}}{\left(a_1^{\dagger\dagger\dagger}b_2^{\dagger\dagger} + \mathrm{i}a_2^{\dagger\dagger}b_1'''\right)c_1''}
\end{aligned}
\tag{86}
$$

etc. Discrete sign choices of square roots come in when expressing the transformed variables, e.g., b_2', a_1''', etc in terms of the original eight variables $a_1, a_2, \dots, d_2$ via (68). For example, from the first lines of (68):

$$
b_2' = \pm\frac{1}{c_2d_2}\sqrt{b_1^2d_2^2 + b_2^2 + c_2^2} \qquad c_2' = \pm\frac{1}{b_2d_2}\sqrt{c_1^2d_2^2 + \left(b_2^2 + c_2^2\right)d_1^2}
$$

$$
d_2' = \pm\frac{1}{b_2c_2}\sqrt{\left(b_2^2 + b_1^2d_2^2\right)c_1^2 + b_1^2c_2^2d_1^2} \qquad a_2'' = \pm\frac{1}{c_1d_1}\sqrt{\left(a_1^2d_1^2 + a_2^2d_2'^2\right)c_2'^2 + c_1^2d_2'^2}.
$$

All primed or daggered variables are square roots of rational expressions.

From (77) the factor ρ is

$$
\rho = \sqrt{\frac{\prod_{i=1}^{4} Z_{12,i}\left(1 + Z_{34,i}\left(Z_{13,i}Z_{14,i}^{-1} - Z_{14,i}Z_{13,i}^{-1}\right) - Z_{34,i}^2\right)}{\prod_{j=5}^{8} Z_{12,j}\left(1 + Z_{34,j}\left(Z_{13,j}Z_{14,j}^{-1} - Z_{14,j}Z_{13,j}^{-1}\right) - Z_{34,j}^2\right)}}.
\tag{87}
$$

Using the components like (82) and inserting there (87) and (86), (68) we get the MTE in terms of our eight parameters $a_1, \dots, d_2$ and 16 choices of the signs of

$$
b_2';\ c_2';\ d_2';\ a_2'';\ c_1'';\ d_1'';\ b_1''';\ d_2''';\quad a_1^{\dagger};\ b_1^{\dagger};\ c_1^{\dagger};\ a_2^{\dagger\dagger};\ b_2^{\dagger\dagger};\ d_1^{\dagger\dagger};\ a_1^{\dagger\dagger\dagger};\ c_2^{\dagger\dagger\dagger}.
\tag{88}
$$

We have confirmed the $N = 2$-MTE numerically for all 256 different components, choosing random complex numbers for the eight continuous variables and random signs for the 16 square roots in the variables (88).

We conclude by mentioning that for $N = 3$ we find that of the 3^{12} components of the MTE for $(R^{(j)})_{i_1,i_2,i_3}^{j_1,j_2,j_3}$ given in (33), 2×3^{11} components are just $0 = 0$, while 3^{11} (not all distinct) equations have non-trivial left- and right-hand sides.

5. Conclusions

In this paper, we study the modified tetrahedron equation (44) in which the Boltzmann weights $R_{i_1,i_2,i_3}^{j_1,j_2,j_3}$ depend on Fermat-curve variables via cyclic weight functions, see (33). The conjugation by the R matrix is a rational automorphism of the ultra-local Weyl algebra at the Nth root of unity. The representation of this automorphism as a functional mapping in the space of the parameters of the ultra-local Weyl algebra allows us to obtain the free parametrization of the MTE. By 'free parametrization' we mean that we leave free that solution of the functional tetrahedron equation (42) with which boundary conditions will be chosen. We express the Fermat-curve variables (and so the Boltzmann weights) in terms of an independent set of eight continuous parameters and specify the 16 phases which can be chosen independently. We derive a general expression for the scalar factor of the MTE. For the simplest non-trivial case $N = 2$ the MTE is written out explicitly. In this case it contains 256 linearly independent components.

The MTE allows us to obtain a wide class of new integrable models. The **R** matrices may be combined into cubic blocks obeying globally the usual TE due to the validity of the local MTEs. The advantage of the free parametrization presented here is that it allows us to get the appropriate parametrization for blocks of any size. This is the subject of a forthcoming paper.

New integrable two-dimensional lattice models with parameters living on higher Riemann surfaces can be obtained from the MTE by a contraction process which has been described in [15].

A further important application of the MTE concerns the following: as usual the TE leads to the commutativity of the layer-to-layer transfer matrices, while the MTE can be used to obtain *exchange relations* for the layer-to-layer transfer matrices. The exchange relations are related to isospectrality deformations and form the basis for a functional Bethe ansatz for three-dimensional integrable spin models, see [18].

Note finally, that starting from the results of this paper, we can consider several limits of the parametrization, which are connected to various degenerations of the weights. The usual tetrahedron equation of [4] follows from the MTE in the special regime when the free parameters $u_j, w_j, j = 1, \ldots, 6$ belong to the submanifold of $\mathbb{C}^{12}$ defined by

$$u_l^N - \mathcal{R}_{i,j,k}^{(f)} \cdot u_l^N = w_l^N - \mathcal{R}_{i,j,k}^{(f)} \cdot w_l^N = 0. \tag{89}$$

This variety may be parametrized in terms of spherical geometry data. For example, $\mathbf{R}_{1,2,3}$ may be associated with the spherical triangle with the dihedral angles $\theta_1, \theta_2, \theta_3$ and

$$\kappa_1^N = \tan^2 \frac{\theta_1}{2} \qquad \kappa_2^N = \cot^2 \frac{\theta_2}{2} \qquad \kappa_3^N = \tan^2 \frac{\theta_3}{2}. \tag{90}$$

Acknowledgments

This work was supported in part by the contract INTAS OPEN 00-00055 and by the Heisenberg–Landau program HLP-2002-11. SP's work was supported in part by the grants CRDF RM1-2334-MO-02, RFBR 01-01-00539 and the grant for support of scientific schools RFBR 00-15-9655. SS's work was supported in part by the grants CRDF RM1-2334-MO-02 and RFBR 01-01-00201.

References

[1] Zamolodchikov A B 1981 *Commun. Math. Phys.* **79** 489

[2] Bazhanov V V and Baxter R J 1992 *J. Stat. Phys.* **69** 453
Bazhanov V V and Baxter R J 1993 *J. Stat. Phys.* **71** 839

[3] Boos H E, Mangazeev V V and Sergeev S M 1995 *Int. J. Mod. Phys.* A **10** 4041

[4] Sergeev S M, Mangazeev V V and Stroganov Yu G 1996 *J. Stat. Phys.* **82** 31

[5] Sergeev S M, Bazhanov V V and Mangazeev V V 1995 Quantum dilogarithm and tetrahedron equation *Preprint* IHEP 95–129 (unpublished)
Sergeev S 1996 Operator solutions of simplex equations *Proc. 10th Int. Conf. on Problems of Quantum Field Theory* ed D V Shirkov, D I Kazakov and A A Vladimirov (Dubna: Publishing Department JINR) p 154

[6] Maillard J-M and Sergeev S M 1997 *Phys. Lett.* B **405** 55

[7] Sergeev S M 1997 On a two dimensional system associated with the complex of the solutions of the tetrahedron equation *Preprint* solv-int/9709013

[8] Sergeev S M 1998 *Phys. Lett.* A **253** 145

[9] Sergeev S M 1999 *J. Phys. A: Math. Gen.* **32** 5639

[10] Sergeev S M 2001 Integrable three dimensional models in wholly discrete space-time *Integrable Structures of Exactly Solvable Two-Dimensional Models etc (NATO Science Series II vol 35)* ed S Pakuliak and G von Gehlen (Dordrecht: Kluwer) pp 293–304

[11] Faddeev L D and Volkov A Yu 1992 *Theor. Math. Phys.* **92** 837

[12] Faddeev L D and Kashaev R M 1994 *Mod. Phys. Lett.* A **9** 427

[13] Bazhanov V V and Reshetikhin N Yu 1995 *J. Phys. A: Math. Gen.* **28** 2217

[14] Bazhanov V V and Reshetikhin N Yu 1995 Chiral Potts model and discrete sine-Gordon model at roots of unity unpublished

[15] Sergeev S M 1995 *Mod. Phys. Lett.* A **12** 1393

[16] Sergeev S M 1999 *Theor. Math. Phys.* **118** 378
Sergeev S M 2000 *Theor. Math. Phys.* **124** 1187

[17] Bazhanov V V and Baxter R J 1993 *J. Stat. Phys.* **71** 839

[18] Sergeev S M 2002 Functional equations and quantum separation of variables for 3D spin models *Preprint* MPI 02-46